

THE ANISOTROPIC WEYL LAW AND GRASSMANNIAN EQUIDISTRIBUTION

ANTONIO DE ROSA AND ARIA HALAVATI

ABSTRACT. We establish a Weyl law for the codimension-one anisotropic volume spectrum on compact Riemannian manifolds. We then use this law to obtain equidistribution in the full Grassmannian bundle while keeping the ambient metric fixed. More precisely, on every closed connected smooth Riemannian manifold of dimension three through six, we construct smooth, embedded hypersurfaces stationary for anisotropies smoothly converging to area. The cumulative sum of their area measures, when lifted to the Grassmannian bundle, converges to the uniform measure.

CONTENTS

1	Introduction	2
1.1	Organization and main ideas of the proof	4
1.2	Notation	7
Part 1.	The anisotropic Weyl law	7
2	Introduction and conventions	7
2.1	Anisotropic integrands	7
2.2	Sweepouts and widths	8
2.3	Approximation results	10
3	The frozen anisotropy Weyl law	11
3.1	Preliminaries	11
3.2	The Lusternik-Schnirelmann inequality	12
3.3	The frozen constant	13
4	The variable anisotropy Weyl law	14
4.1	The lower bound	14
4.2	An equivariant cube to slab map	16
4.3	Competitor sweepouts In a long rectangle	22
4.4	The upper bound	26
Part 2.	Grassmannian equidistribution for nearby anisotropies	29
5	The smooth realization assumption and the conclusions	29
6	Averaged derivatives of the Weyl functional	30
6.1	Finite-dimensional families of test functions	30
6.2	Averaging the Weyl derivatives	31
6.3	Passing to the volume spectrum	32
7	Geometric realization and equidistribution	33
7.1	Deformations of stationary hypersurfaces	33
7.2	Derivatives of the widths	36
7.3	Finite stationary collections	37
7.4	Proof of equidistribution	40
Part 3.	Smooth realization of the anisotropic widths	40
8	Introduction and preliminaries	40

8.1	The setup	41
8.2	Main results	42
9	Almost minimizing varifolds	43
9.1	Pull tight	43
9.2	Curvature estimates and regularity of replacements	45
9.3	Smooth gluing of consecutive replacements	49
10	Removing singularities	51
	References	56

1. INTRODUCTION

The existence and distribution of closed minimal hypersurfaces are central questions in geometric analysis. The topology of the space of cycles, identified by Almgren [3], provides the basis for a variational approach to these questions. Combined with the min-max construction of Pitts [42] and the regularity theory of Schoen-Simon [45], it produces smooth closed embedded minimal hypersurfaces in every closed Riemannian manifold of dimension between three and seven. Multiparameter families of cycles encode further topological information and give rise to a sequence of variational invariants, the volume spectrum. For a bumpy metric on a closed manifold of dimension three through seven, Zhou [55] proved that every width in the volume spectrum admits a realization by a disjoint union of smooth, closed, embedded, two-sided minimal hypersurfaces, each with multiplicity one. Such metrics form a residual set in the space of smooth Riemannian metrics by White's bumpy metrics theorem [51, 52].

The asymptotic behavior of this spectrum connects min-max theory with the global geometry of the ambient manifold. The estimates of Gromov [28] and Guth [29] show that the p -width of a fixed compact d -dimensional manifold grows at the rate $p^{1/d}$. Liokumovich-Marques-Neves [37] identified the leading coefficient by proving the Weyl law

$$\lim_{p \rightarrow \infty} p^{-1/d} \omega_p^{d-1}(M, 1) = a_d \operatorname{Vol}_g(M)^{(d-1)/d}, \quad (1.1)$$

where $a_d > 0$ depends only on the dimension. Thus the large-index behavior of a nonlinear variational spectrum recovers the volume of the manifold, in analogy with the classical Weyl law for eigenvalues.

These ideas have led to substantial progress on the abundance of minimal hypersurfaces. In particular Yau [54, Problem 88] conjectured that every closed Riemannian three-manifold contains infinitely many smooth, closed, immersed minimal surfaces. Marques-Neves [38] proved the existence of infinitely many such hypersurfaces under positive Ricci curvature, and Song [49] established the corresponding result for arbitrary closed metrics in dimensions three through seven (see also [35, 13, 14]). The Weyl law also makes it possible to address their distribution. Irie-Marques-Neves [32] proved that, for a C^∞ -generic metric in these dimensions, the union of the closed smooth embedded minimal hypersurfaces is dense. Subsequently, Marques-Neves-Song [40] constructed a sequence whose cumulative area measures equidistribute with respect to normalized Riemannian volume. Their theorem also controls tangent-plane averages tested against traces of symmetric two-tensors. Equidistribution coexists with concentration phenomena. Song-Zhou [50] proved that, for a C^∞ -generic metric in dimensions three through seven, every connected, closed, embedded, two-sided, stable minimal hypersurface is approached by a sequence of closed embedded minimal hypersurfaces whose areas diverge and whose normalized varifolds converge to its normalized varifold.

The natural next question concerns the simultaneous distribution of positions and tangent planes. Write $G_{d-1}(M)$ for the bundle of unoriented hyperplanes, and let σ_x be

the invariant probability measure on $G_{d-1}(T_x M)$. Full Grassmannian equidistribution asks for convergence against every continuous function on $G_{d-1}(M)$, with limiting probability measure

$$\int_{G_{d-1}(M)} h d\Lambda_g := \frac{1}{\text{Vol}_g(M)} \int_M \int_{G_{d-1}(T_x M)} h(x, T) d\sigma_x(T) dV_g(x). \quad (1.2)$$

The question of extending equidistribution to the Grassmannian bundle is explicitly raised by Marques-Neves-Song in [40, Introduction, final paragraph]. It asks for substantially more than the tensor-trace conclusion: in codimension one, $\text{tr}_T k = \text{tr}_g k - k(\nu, \nu)$ for a unit normal ν , so those tests see only constant and quadratic angular components. A general continuous function on the hyperplane bundle can contain angular components of arbitrarily high degree.

In this paper, we develop an anisotropic approach to this problem. We keep the Riemannian metric fixed and allow the surface energy to depend on the tangent plane. More precisely, for a positive function $F : G_{d-1}(TM) \rightarrow (0, \infty)$, we consider

$$\mathcal{A}_F(\Sigma) = \int_{\Sigma} F(x, T_x \Sigma) dA_g.$$

The area functional corresponds to $F \equiv 1$. The integrand F can be also seen as the positively one-homogeneous integrand defined on the tangent bundle $\varphi_x(\xi) = |\xi|_g F(x, \xi^\perp)$ for $x \in M$ and $\xi \neq 0 \in T_x M$, with $\varphi_x(0) = 0$. Convexity of this extension is the natural assumption. Varying F permits arbitrary smooth angular tests, while retaining the background metric and the measure (1.2).

Our first main result is a Weyl law for the anisotropic volume spectrum. We use $\mathbb{Z}_2$ coefficients, relative cycles when the ambient manifold has boundary, and sweepouts continuous in the strong mass norm. First we define the Weyl constant for frozen anisotropies and prove that the limit exists:

Theorem 1.1. *Let H be an autonomous integrand satisfying H1-H3. The limit*

$$a(H) := \lim_{p \rightarrow \infty} p^{-1/d} \omega_p^{d-1}((0, 1)^d, H) = \sup_{q \geq 1} q^{-1/d} \omega_q^{d-1}((0, 1)^d, H). \quad (1.3)$$

exists and belongs to $(0, \infty)$. We call $a(H)$ the frozen Weyl constant associated with H .

Moreover, for every bounded connected Lipschitz domain $\Omega \subset \mathbb{R}^d$, we have

$$\lim_{p \rightarrow \infty} p^{-1/d} \omega_p^{d-1}(\Omega, H) = a(H) |\Omega|^{(d-1)/d}. \quad (1.4)$$

Rotation invariance of a allows us to compute $a(F(x, \cdot))$ in any orthonormal frame at x . The resulting coefficient is intrinsic. The following theorem extends the frozen Weyl law to variable integrands on manifolds. Both results are proved in Part 1.

Theorem 1.2. *Let (M^d, g) be compact, connected, and smooth, with possibly empty smooth boundary, and let $d \geq 2$. Assume that F satisfies H1-H3. Then*

$$\lim_{p \rightarrow \infty} p^{-1/d} \omega_p^{d-1}(M, F) = \mathcal{W}(F) := \left(\int_M a(F(x, \cdot))^{d/(d-1)} dV_g(x) \right)^{(d-1)/d}. \quad (1.5)$$

In the isotropic case, (1.5) reduces to (1.1).

The second main result concerns the distribution of stationary hypersurfaces. Part 2 proves this conclusion under the smooth realization assumption, which is established in Part 3.

Theorem 1.3. *Let (M^d, g) be a closed connected smooth Riemannian manifold, with $3 \leq d \leq 6$, and let Λ_g be the probability measure defined in (1.2). There exist smooth*

positive strictly elliptic anisotropies $F_j \rightarrow 1$ in $C^\infty(G_{d-1}(TM))$, and smooth, closed, embedded, connected F_j -stationary hypersurfaces Σ_j such that

$$\lim_{q \rightarrow \infty} \frac{\sum_{j=1}^q \int_{\Sigma_j} h(x, T_x \Sigma_j) dA_g}{\sum_{j=1}^q \text{area}_g(\Sigma_j)} = \int_{G_{d-1}(TM)} h d\Lambda_g \quad \text{for every } h \in C^0(G_{d-1}(TM)).$$

For every j , there is an integer $p_j \geq 1$ such that Σ_j is a nondegenerate component of a smooth varifold realizing $\omega_{p_j}^{d-1}(M, F_j)$.

Thus we obtain full Grassmannian equidistribution through hypersurfaces stationary for energies converging smoothly to area. This is an anisotropic approximation statement in the direction of the question raised by Marques-Neves-Song in [40, Introduction, final paragraph]. The hypersurfaces Σ_j are stationary for their respective integrands F_j ; they are not asserted to be minimal for area. On the other hand, the conclusion holds for every fixed metric g , without a genericity assumption on the metric.

Part 3 proves a key tool to achieve Theorem 1.3: the smooth realization of any width.

Theorem 1.4. *For every fixed closed smooth (M^d, g) , $3 \leq d \leq 6$, and every fixed integer $k_0 \geq 6$, there exists $\delta_0 > 0$ such that every smooth $F: G_{d-1}(TM) \rightarrow (0, \infty)$ with $\|F - 1\|_{C^{k_0}} < \delta_0$ and every $p \geq 1$ admit an integral varifold*

$$V = \sum_{i=1}^s m_i |\Sigma_i|, \quad \int F dV = \omega_p^{d-1}(M, F),$$

where $m_i \in \mathbb{N}$ and the Σ_i are smooth closed embedded connected pairwise disjoint F -stationary hypersurfaces. δ_0 is indeed independent of p .

The smooth realization requires a regularity theory adapted to anisotropic min-max hypersurfaces. This subject presents additional difficulties even for smooth uniformly elliptic integrands. The general minimizing regularity theory of Schoen-Simon-Almgren [46] gives weaker bounds on the singular set than the corresponding theory for area, and the usual area monotonicity formula is unavailable for a general anisotropic stationary varifold [1]. The lack of monotonicity is a serious obstruction to the development of the anisotropic minimal surface theory [19, 20]. De Philippis-De Rosa [16] developed a min-max construction of anisotropic minimal and CMC surfaces in 3-manifolds. In the isotropic case, the min-max theory for CMC hypersurfaces was previously developed by Zhou-Zhu [56] in closed Riemannian manifolds of dimension three through seven. De Philippis-De Rosa-Li [18] subsequently established existence and regularity results in arbitrary dimension. A complementary approach through anisotropic Allen-Cahn energies was developed by De Rosa-Pigati [22], producing possibly singular stationary integral varifolds. The multiplicity of these varifolds has been analyzed in the recent work [8]. More recently, De Rosa-Halavati-Wang [21] established anisotropic minimization in isotopy classes and a Simon-Smith construction with genus control, together with isolated-singularity removal under pinching assumptions. Their removable-singularity theorem also applies to the 3-dimensional multiparameter Almgren-Pitts construction.

We prove that every width on a fixed closed d -manifold, $3 \leq d \leq 6$ is realized by a smooth embedded stationary varifold throughout one neighborhood of area, with the neighborhood independent of the index p ; see Theorem 1.4. To this aim, we establish near-area minimizing regularity, stable curvature estimates, and isolated-singularity removal.

1.1. Organization and main ideas of the proof. The paper is organized in three parts. Part 1 determines the asymptotic behavior of the anisotropic widths. Part 2

converts this spectral information into equidistribution of positions and tangent planes in the Grassmannian bundle, assuming that the widths are attained by finite unions of smooth stationary hypersurfaces with positive integer multiplicities. Part 3 establishes this smooth realization property in dimension three through six. Thus the Weyl law, together with smooth realization, yields the equidistribution theorem. We explain below the proof strategy of three parts.

The anisotropic Weyl law. The first task is to set up a class of sweepouts that can be localized and assembled without losing control of either their topology or their anisotropic energy. We work with relative $\mathbb{Z}_2$ cycles and sweepouts continuous in the strong mass topology. The approximation theorem in Part 1 converts the flat families arising in geometric constructions into admissible strong mass families, with arbitrarily small increase of the maximal anisotropic energy.

The lower bound follows from localization and the Lusternik-Schnirelmann inequality. In codimension one, the two complementary fillings of a relative cycle form a double cover encoding the sweepout class. Slicing and deformation give restriction maps to the chosen subdomains with arbitrarily small errors, while preserving the relevant cohomology class. The relative cup-product argument then bounds a global width from below by a sum of local widths.

For an autonomous integrand H , tiling the unit cube and using homogeneity give the existence of the frozen coefficient $a(H)$. Packing by translated and dilated copies of domains then proves that the same coefficient governs every bounded connected Lipschitz domain. This part follows the self-similar strategy of [37].

For a variable integrand, we freeze the coefficients on small coordinate cubes. In the lower bound, the asymptotically optimal allocation of the index is

$$\frac{p_i}{p} \simeq \frac{|C_i| a(H_i)^{d/(d-1)}}{\sum_j |C_j| a(H_j)^{d/(d-1)}}.$$

This gives the integral and the exponent appearing in $\mathcal{W}(F)$.

The upper bound requires a different construction. We first transform filling families on cubes into families on slabs, using a homothety followed by cuts and translations. The homothety produces the required scaling factor, and the cuts contribute an explicitly controlled additive error. No rotation or nonuniform linear stretching is used.

We then combine the local families through a product and join of their filling double covers. The cohomological height of the product is controlled by the smallest local height, while the join adds these heights. The join parameters distribute the available slab volumes, and maximizing the resulting energy expression produces the same coefficient as the lower-bound allocation. Finally, localization in coordinate charts and control of the additional interfaces give the upper bound on the manifold.

From spectral asymptotics to equidistribution. Part 2 is devoted to the proof of the equidistribution Theorem 1.3, under the assumption of the smooth realization property of the width, stated in Assumption 5.1. We aim to represent spectral derivatives by integrals over smooth stationary hypersurfaces. We vary the anisotropic integrand, while keeping the metric fixed. A perturbation by a function $h(x, T)$ tests the distribution of both positions and tangent planes, making full Grassmannian equidistribution accessible.

There are two difficulties in extracting this information from the Weyl law: the frozen coefficient $a(F(x, \cdot))$ is not explicitly known, and we do not assume its differentiability at the area integrand. We therefore work with averaged derivatives in finite-dimensional families of anisotropies. Rotation invariance identifies the angular average, and homogeneity determines its normalization. Finite spaces of fiberwise angular functions are assembled using Parseval frames, so the averaging construction is defined globally even when the tangent bundle is nontrivial. As the perturbations shrink to area, these averaged derivatives

recover the functional associated with Riemannian volume and the invariant probability measure on each Grassmannian fiber.

To relate this functional to hypersurfaces, we establish an anisotropic deformation argument with the metric fixed. After a small generic perturbation of a finite-parameter family, all stationary hypersurfaces are nondegenerate at almost every parameter. The local deformation theorem and countability then give countably many C^1 branches of connected stationary hypersurfaces. Finite collections of these branches, with positive integer multiplicities, describe the smooth realizations of the widths. A countable selection argument identifies the differential of each width almost everywhere with the differential of the energy of one such collection. More precisely, at such a parameter there is a smooth realizing varifold $V_{p,t}$ for which

$$D_t \omega_p^{d-1}(M, F(t))[v] = \int_{G_{d-1}(TM)} D_t F(t)[v](x, T) dV_{p,t}(x, T).$$

The contribution from the motion of the hypersurface vanishes by stationarity. This realizes all directions in the chosen parameter space with the same varifold and avoids a compactness limit of derivative-realizing hypersurfaces.

The Weyl law gives locally uniform convergence of the normalized widths in each finite-dimensional family. To pass to derivatives, we integrate by parts against a smooth parameter density. Thus we pass to the limit in averaged derivatives without claiming pointwise convergence of derivatives. We first fix the perturbation scale and let $p \rightarrow \infty$, and only afterwards let the scale tend to zero. The generic perturbation of the parameter family is chosen small enough in C^1 that its effect on both energies and parameter derivatives is negligible at that scale. Combining these facts with the preceding realization formula produces finite positive weighted collections of smooth stationary hypersurfaces whose moments approximate those of the target Grassmannian measure. We impose increasingly many test functions and make the anisotropies converge to area in increasingly strong smooth norms. Rational approximation of the weights and a careful repetition of the resulting finite lists then produce one sequence whose normalized cumulative area measures converge as the number of terms tends to infinity.

Smooth realization and the regularity arguments. Part 3 proves smooth realization for finite-parameter Almgren-Pitts constructions in ambient dimensions $3 \leq d \leq 6$. Pull-tight and the almost-minimizing construction are performed without assuming an upper density bound.

First we derive an area-independent curvature estimate for stable hypersurfaces when the integrand is sufficiently close to area. A blow-up argument reduces this estimate to the near-area Bernstein theorems in Euclidean dimensions three through six. Together with near-area minimizing regularity, it gives smooth compactness of replacements. A boundary multiplicity argument in strictly convex lenses then permits successive replacements to be glued smoothly, yielding regularity away from finitely many points.

To remove these remaining points, we adapt the separation function of [21, Section 6]. A strict radial Jacobi barrier gives a linear lower bound for the separation between neighboring sheets. This yields the density bound needed to pass stationarity and stability across the puncture. In hypersurface dimension two, finite total curvature allows the use of White's removable-singularity theorem. In dimensions three through five, an autonomous blow-up argument and a critical curvature estimate exclude curvature concentration at the puncture. All smallness thresholds for the integrand are independent of the width index and the number of parameters.

Use of LLMs. The ideas in sections 4.2, 4.3 were discovered with the help of ASTRA, in particular the initial cube to slab map and the idea of taking higher order joins were LLM generated, which helped close the arguments in the upper bound for the Weyl law. We also

used ASTRA to improve the exposition of the article, including writing help, generating figures and speeding up computations.

Acknowledgement. The authors were funded by the European Union: the European Research Council (ERC), through StG “ANGEVA”, project number: 101076411. Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them.

1.2. Notation. The ambient dimension is denoted by d , and all cycles and hypersurfaces have dimension $d - 1$. We write

$$G := G_{d-1}(TM).$$

For $U \subset M$, we use

$$G_{d-1}(U) := \{(x, T) \in G : x \in U\};$$

in particular, $G_{d-1}(M) = G$. We write dV_g and dA_g for Riemannian volume and hypersurface area.

Unless otherwise stated, currents and cohomology groups have coefficients in $\mathbb{Z}_2$. For a current T , its ordinary mass, mass measure, and associated varifold are denoted by $\mathbf{M}(T)$, $\|T\|$, and $|T|$, respectively. We use $\mathcal{F}$ for flat distance. Relative cycles, their canonical representatives, and the sweepout class are defined in Section 2.2.

The symbol $\mathbf{F}$ denotes anisotropic energy, whereas $\mathbb{F}$ denotes the varifold metric and the corresponding metric on cycles used in Part 3. The normal representation of an integrand and the anisotropic perimeter are defined below. The quantities a , $\mathcal{W}$, and Λ_g retain the meanings given in the introduction.

Part 1. The anisotropic Weyl law

2. INTRODUCTION AND CONVENTIONS

Throughout, $d \geq 2$ is the ambient dimension, and all cycles have dimension $d - 1$. We use the notation

$$\tilde{\omega}_p^{d-1}(M, F) = p^{-1/d} \omega_p^{d-1}(M, F).$$

In this article we only consider the case of co-dimension one.

2.1. Anisotropic integrands. Let (M^d, g) be compact, smooth, and connected, with smooth boundary (possibly empty). Let $G_{d-1}(M)$ be the bundle of unoriented tangent $(d - 1)$ -planes:

$$G_{d-1}(M) = \{(x, T) : x \in M, T \subset T_x M \text{ is a } (d - 1)\text{-dimensional linear subspace}\},$$

and let $F : G_{d-1}(TM) \rightarrow (0, \infty)$ be an anisotropic integrand. For a rectifiable varifold V and an open set U write

$$\mathbf{F}(V, U) = \int_{G_{d-1}(U)} F(x, T) dV(x, T).$$

The normal representation of F is

$$\varphi_x(\xi) = |\xi|_g F(x, \xi^\perp) \quad (\xi \neq 0), \quad \varphi_x(0) = 0.$$

It is even and positively one-homogeneous. We assume the following conditions on the anisotropy throughout the proof of this part:

- (H1) (Comparability) There is $0 < \lambda \leq 1$ such that $\lambda \leq F(x, T) \leq \lambda^{-1}$.
- (H2) (Regularity) F is continuous on $G_{d-1}(M)$.
- (H3) (Ellipticity) For each x , φ_x is convex.

In particular these assumptions hold under the smooth uniformly elliptic hypotheses used for smooth integrands: $F \in C^3$, a bounded C^3 norm, and

$$D_{\xi\xi}^2 \varphi_x(\xi)[\zeta, \zeta] \geq \lambda \frac{|\zeta|_g^2}{|\xi|_g} \quad \text{if } \xi \neq 0, \quad \langle \zeta, \xi \rangle_g = 0.$$

Convexity follows from this inequality and homogeneity. We will frequently also denote an autonomous integrand on the Euclidean space with H .

For a finite-perimeter set E in U , define

$$P_F(E; U) := \int_{\partial^* E \cap U} F(x, \nu_E^\perp) d\mathcal{H}^{d-1}, \quad P(E; U) := \mathcal{H}^{d-1}(\partial^* E \cap U).$$

We write $\mathcal{C}(U)$ for the finite-perimeter sets in U , identified modulo null sets and equipped with the L^1 topology. Since φ_x is even, the anisotropic perimeter is independent of the sign chosen for the normal.

2.2. Sweepouts and widths. We follow the conventions of [37], restricting throughout to codimension one and coefficients in $\mathbb{Z}_2$. For the relative-cycle framework and the corresponding free-boundary min-max theory for the area functional, see also Li-Zhou [34]. We briefly recall the notation.

For $m \in \mathbb{N}$, we set $I^m = [0, 1]^m$. For each integer $j \geq 0$, we define $I(1, j)$ to be the cubical complex with vertices $\{i3^{-j} : 0 \leq i \leq 3^j\}$, and one-dimensional cells

$$[i3^{-j}, (i+1)3^{-j}], \quad 0 \leq i < 3^j.$$

We define

$$I(m, j) = I(1, j)^{\otimes m}.$$

A cell $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_m$ has dimension $\sum_{i=1}^m \dim \alpha_i$, and we identify it with its support $\alpha_1 \times \cdots \times \alpha_m \subset I^m$. If X is a cubical subcomplex of $I(m, j_0)$, we define $X(j)$, for $j \geq j_0$, to consist of the cells of $I(m, j)$ contained in X . We denote its collection of q -cells by $X(j)_q$. We identify each cubical complex with its underlying topological space.

We fix an isometric embedding of M into some Euclidean space $\mathbb{R}^Q$. For $0 \leq k \leq d$, we denote by $\mathcal{R}_k(M; \mathbb{Z}_2)$ the space of k -dimensional rectifiable flat chains with coefficients in $\mathbb{Z}_2$ supported in M . For $k \geq 1$, we set

$$\mathbf{I}_k(M; \mathbb{Z}_2) := \{T \in \mathcal{R}_k(M; \mathbb{Z}_2) : \partial T \in \mathcal{R}_{k-1}(M; \mathbb{Z}_2)\}.$$

We use the analogous notation for chains supported in ∂M . For a rectifiable chain T , we denote its mass measure by $\|T\|$, its mass by $\mathbf{M}(T)$, and its support by $\text{spt}(T)$. In some parts of the paper we also use the notation $\mathbf{M}_F$ to mean the anisotropic mass of a current with respect to F . For $S, T \in \mathcal{R}_{d-1}(M; \mathbb{Z}_2)$, we define the flat distance by

$$\mathcal{F}(S, T) := \inf \{\mathbf{M}(R) + \mathbf{M}(Q) : T - S = R + \partial Q, R \in \mathcal{R}_{d-1}(M; \mathbb{Z}_2), Q \in \mathcal{R}_d(M; \mathbb{Z}_2)\}.$$

We define

$$\mathcal{Z}_{d-1}(M, \partial M; \mathbb{Z}_2) := \{T \in \mathbf{I}_{d-1}(M; \mathbb{Z}_2) : \text{spt}(\partial T) \subset \partial M\}.$$

We identify two elements S, T of this space when

$$S - T \in \mathbf{I}_{d-1}(\partial M; \mathbb{Z}_2),$$

and we denote the resulting quotient by $\mathcal{Z}_{d-1, \text{rel}}(M, \partial M; \mathbb{Z}_2)$. We write

$$P : \mathcal{Z}_{d-1}(M, \partial M; \mathbb{Z}_2) \longrightarrow \mathcal{Z}_{d-1, \text{rel}}(M, \partial M; \mathbb{Z}_2)$$

for the quotient map. When $\partial M = \emptyset$, we recover the usual space $\mathcal{Z}_{d-1}(M; \mathbb{Z}_2)$ of cycles.

We define the mass and flat distance of relative cycles by

$$\begin{aligned}\mathbf{M}(P(T)) &:= \inf_{R \in \mathbf{I}_{d-1}(\partial M; \mathbb{Z}_2)} \mathbf{M}(T + R), \\ \mathcal{F}(P(S), P(T)) &:= \inf_{R \in \mathbf{I}_{d-1}(\partial M; \mathbb{Z}_2)} \mathcal{F}(S + R, T).\end{aligned}$$

We equip the relative cycle space either with this flat distance or with the strong mass distance $(\sigma, \tau) \mapsto \mathbf{M}(\sigma - \tau)$. For a relative class $\tau = P(T)$, we define its mass measure by

$$\|\tau\|(A) := \|T\|(A \setminus \partial M) \quad \text{for every Borel set } A \subset M.$$

This definition does not depend on the representative, and [37, Proposition 2.4] gives $\mathbf{M}(\tau) = \|\tau\|(M)$. Thus we do not count mass supported on the boundary.

For $T \in \mathcal{R}_{d-1}(M; \mathbb{Z}_2)$, we denote by $|T|$ its associated rectifiable varifold and, consistently with the preceding subsection, set

$$\mathbf{F}(T) := \mathbf{F}(|T|, M).$$

We define the anisotropic mass of a relative cycle by

$$\mathbf{F}(P(T)) := \inf_{R \in \mathbf{I}_{d-1}(\partial M; \mathbb{Z}_2)} \mathbf{F}(T + R).$$

Equivalently, we integrate only over the interior:

$$\mathbf{F}(P(T)) = \mathbf{F}(|T|, M \setminus \partial M).$$

Indeed, representatives agree away from ∂M , and the boundary approximation used in [37, Proposition 2.4], together with (H1), removes the boundary contribution with arbitrarily small anisotropic cost. In particular, for relative cycles σ, τ , we have

$$\lambda \mathbf{M}(\tau) \leq \mathbf{F}(\tau) \leq \lambda^{-1} \mathbf{M}(\tau), \quad |\mathbf{F}(\sigma) - \mathbf{F}(\tau)| \leq \lambda^{-1} \mathbf{M}(\sigma - \tau). \quad (2.1)$$

We denote by $\mathcal{Z}_{d-1, \text{rel}}^0(M, \partial M; \mathbb{Z}_2)$ the path component of zero in the flat topology. Since M is connected, Almgren's isomorphism [3] gives

$$\pi_1(\mathcal{Z}_{d-1, \text{rel}}^0(M, \partial M; \mathbb{Z}_2), 0) \simeq H_d(M, \partial M; \mathbb{Z}_2) \simeq \mathbb{Z}_2.$$

We denote by $\bar{\lambda}$ the unique nonzero element of $H^1(\mathcal{Z}_{d-1, \text{rel}}^0(M, \partial M; \mathbb{Z}_2); \mathbb{Z}_2)$, where we compute cohomology using the flat topology.

Let $p \in \mathbb{N}$ and let X be a finite cubical subcomplex of some $I(m, j)$. We call a flat-continuous map

$$\Phi : X \longrightarrow \mathcal{Z}_{d-1, \text{rel}}^0(M, \partial M; \mathbb{Z}_2)$$

a *p-sweepout* if

$$\Phi^*(\bar{\lambda}^p) = (\Phi^* \bar{\lambda})^p \neq 0 \quad \text{in } H^p(X; \mathbb{Z}_2).$$

This condition depends only on the flat homotopy class of Φ .

Throughout this paper, we use sweepouts continuous in the strong mass topology. We denote by $\mathcal{P}_p^{d-1}(M)$ the collection of all such p -sweepouts, allowing both the parameter complex X and its ambient dimension to vary. Every map in this collection has no concentration of mass by [37, Lemma 2.7]; explicitly,

$$\limsup_{\substack{r \downarrow 0 \\ x \in X \\ y \in M}} \|\Phi(x)\|(B_r^g(y)) = 0, \quad (2.2)$$

where $B_r^g(y)$ denotes the ball in M for the Riemannian distance. By H1, this condition also excludes concentration of anisotropic mass.

We define the anisotropic p -width by

$$\omega_p^{d-1}(M, F) := \inf_{\Phi \in \mathcal{P}_p^{d-1}(M)} \sup_{x \in \text{dmn}(\Phi)} \mathbf{F}(\Phi(x)).$$

We retain the normalization

$$\tilde{\omega}_p^{d-1}(M, F) := p^{-1/d} \omega_p^{d-1}(M, F).$$

2.3. Approximation results. We follow [37, Section 2.9] and explain the modification needed to control the anisotropic energy.

In our later constructions, we will obtain families by restricting and assembling local sweepouts. These operations naturally produce maps continuous in the flat topology. The next proposition allows us to work with such families, provided we verify that they have no concentration of mass. We can then recover strong mass continuity on the same parameter space, preserve the sweepout class, and increase the maximal anisotropic energy by an arbitrarily small amount.

Proposition 2.1. *Let F satisfy (H1-H3), let X be a finite cubical subcomplex of some $I(m, j_0)$, and let*

$$\Phi : X \longrightarrow \mathcal{Z}_{d-1, \text{rel}}^0(M, \partial M; \mathbb{Z}_2)$$

be continuous in the flat topology and have no concentration of mass, in the sense of (2.2). For every $\varepsilon > 0$, there exists a map

$$\Phi_\varepsilon : X \longrightarrow \mathcal{Z}_{d-1, \text{rel}}^0(M, \partial M; \mathbb{Z}_2)$$

continuous in the strong mass topology and homotopic to Φ in the flat topology, such that

$$\begin{aligned} \sup_{x \in X} \mathcal{F}(\Phi_\varepsilon(x), \Phi(x)) &< \varepsilon, \\ \sup_{x \in X} \mathbf{F}(\Phi_\varepsilon(x)) &\leq \sup_{x \in X} \mathbf{F}(\Phi(x)) + \varepsilon. \end{aligned}$$

In particular, if Φ is a flat-continuous p -sweepout, then $\Phi_\varepsilon \in \mathcal{P}_p^{d-1}(M)$.

Proof. We follow [37, Theorems 2.10-2.11 and Proposition 2.12]. In the discretization argument of Appendix A, local lower semicontinuity of the anisotropic perimeter [44] gives the energy comparison for the replaced portions, while comparability controls the added slicing chains by their ordinary mass. Thus the same construction, with smaller error tolerances, preserves the anisotropic energy bound and the mass-fineness estimates. The Almgren extension preserves this bound by (2.1). Uniform flat approximation and flat homotopy follow from the cited results, and homotopy invariance preserves the p -sweepout condition. $\square$

We next record the no-concentration property of mass-continuous families, following [37, Lemma 2.7]. This shows that our admissible sweepouts automatically satisfy the additional hypothesis in Proposition 2.1.

Proposition 2.2. *Let X be a finite cubical complex and let*

$$\Phi : X \longrightarrow \mathcal{Z}_{d-1, \text{rel}}^0(M, \partial M; \mathbb{Z}_2)$$

be continuous in the strong mass topology. Then

$$\lim_{r \downarrow 0} \sup_{\substack{x \in X \\ y \in M}} \|\Phi(x)\|(B_r^g(y)) = 0.$$

In particular, Φ has no concentration of mass or anisotropic mass.

Proof. We fix $\varepsilon > 0$. By compactness of X and mass continuity, we choose $x_1, \dots, x_N \in X$ such that, for every $x \in X$, there exists i with

$$\mathbf{M}(\Phi(x) - \Phi(x_i)) < \frac{\varepsilon}{2}.$$

Since $d - 1 \geq 1$, each rectifiable mass measure $\|\Phi(x_i)\|$ has no atoms. Compactness of M then gives

$$\lim_{r \downarrow 0} \sup_{y \in M} \|\Phi(x_i)\|(B_r^g(y)) = 0.$$

Indeed, a sequence of shrinking balls with masses bounded away from zero would, after taking a convergent subsequence of their centers, produce an atom.

Since we have only finitely many measures, we choose $r > 0$ such that

$$\sup_{y \in M} \|\Phi(x_i)\|(B_r^g(y)) < \frac{\varepsilon}{2} \quad \text{for } i = 1, \dots, N.$$

For $x \in X$, we choose i as above and obtain, for every $y \in M$,

$$\|\Phi(x)\|(B_r^g(y)) \leq \|\Phi(x_i)\|(B_r^g(y)) + \mathbf{M}(\Phi(x) - \Phi(x_i)) < \varepsilon.$$

This proves the assertion for mass. The corresponding statement for anisotropic mass follows from $F \leq \lambda^{-1}$. $\square$

Combining the two propositions, we can compute $\omega_p^{d-1}(M, F)$ using flat-continuous p -sweepouts with no concentration of mass. Indeed, every admissible mass-continuous sweepout belongs to this larger class, and Proposition 2.1 gives the reverse inequality between the two infima by letting $\varepsilon \downarrow 0$.

3. THE FROZEN ANISOTROPY WEYL LAW

The frozen Weyl law follows similarly to the packing and self-similar strategy of [37]. Here we outline the main adaptations to anisotropies.

3.1. Preliminaries. We first compare the anisotropic widths with the ordinary volume spectrum. We then establish the anisotropic version of the restriction estimates in [37, Section 2.14 and Lemma 2.15], following the same construction and explaining the modification needed for the energy estimate.

Lemma 3.1. *Let (M^d, g) be as in the preceding section, and let F satisfy H1. There exists $C_{\lambda, d, M} > 1$ such that*

$$C_{\lambda, d, M}^{-1} \leq \tilde{\omega}_p^{d-1}(M, F) \leq C_{\lambda, d, M} \quad \text{for every } p \geq 1.$$

The same conclusion holds when M is a bounded connected Lipschitz domain in $\mathbb{R}^d$, with widths computed relative to its boundary.

Proof. For every admissible sweepout Φ , we have

$$\lambda \sup_{x \in \text{dmn}(\Phi)} \mathbf{M}(\Phi(x)) \leq \sup_{x \in \text{dmn}(\Phi)} \mathbf{F}(\Phi(x)) \leq \lambda^{-1} \sup_{x \in \text{dmn}(\Phi)} \mathbf{M}(\Phi(x)).$$

Taking the infimum over the same collection of sweepouts gives

$$\lambda \omega_p^{d-1}(M, 1) \leq \omega_p^{d-1}(M, F) \leq \lambda^{-1} \omega_p^{d-1}(M, 1).$$

We multiply by $p^{-1/d}$ and apply the Gromov-Guth bounds [28, 29]; see also [37, Theorem 2.8]. $\square$

For the remainder of this subsection, we fix an autonomous integrand H satisfying (H1-H3), and we use $\mathbf{F}$ for its associated energy. We consider bounded connected Lipschitz domains in $\mathbb{R}^d$ and compute their widths relative to their boundaries.

Let $R \subset \Omega$ be two such domains. For a relative cycle $\tau = P(T)$ in Ω , we define

$$\mathbf{F}(\tau \llcorner R) := \mathbf{F}(|T|, R).$$

This quantity does not depend on the representative. We integrate over the open domain R , excluding energy supported on ∂R . The chain $T \llcorner R$ need not have finite boundary mass, so it need not itself define an integral relative cycle in R . We address this issue using the restriction construction of [37, Lemma 2.15].

Lemma 3.2. *Let $R \subset \Omega$ be bounded connected Lipschitz domains in $\mathbb{R}^d$, let $p \geq 1$, and let $\Phi \in \mathcal{P}_p^{d-1}(\Omega)$ with $X = \text{dmn}(\Phi)$. Then the following statements hold.*

(i) For every $\varepsilon > 0$, there exists a map $\widehat{\Phi} : X \rightarrow \mathcal{Z}_{d-1,\text{rel}}^0(R, \partial R; \mathbb{Z}_2)$ continuous in the strong mass topology such that

$$\mathbf{F}(\widehat{\Phi}(x)) \leq (1 + \varepsilon)^{d-1} \mathbf{F}(\Phi(x) \llcorner R) \quad \text{for every } x \in X, \quad (3.1)$$

and

$$\widehat{\Phi}^* \bar{\lambda} = \Phi^* \bar{\lambda} \quad \text{in } H^1(X; \mathbb{Z}_2). \quad (3.2)$$

In (3.2), we use $\bar{\lambda}$ for the distinguished class of the corresponding target cycle space.

(ii) For every $\eta > 0$ and every integer $1 \leq q \leq p$, the set

$$\{x \in X : \mathbf{F}(\Phi(x) \llcorner R) \leq \omega_q^{d-1}(R, H) - \eta\}$$

is contained in an open set $U \subset X$ such that, writing $\iota : U \hookrightarrow X$ for the inclusion,

$$(\iota^* \Phi^* \bar{\lambda})^q = 0 \quad \text{in } H^q(U; \mathbb{Z}_2).$$

Proof. We follow [37, Lemma 2.15]. Its inward deformations f_t converge to the identity in C^1 . By continuity and positivity of H , for all sufficiently small $t > 0$ we have

$$\mathbf{F}((f_t^{-1})_{\#} S) \leq (1 + \varepsilon)^{d-1} \mathbf{F}(S), \quad \text{spt } S \subset f_t(R).$$

The slicing and mass-continuity estimates are unchanged, while lower semicontinuity of the anisotropic energy gives (3.1) in the limiting construction. The same Almgren-isomorphism argument gives (3.2), proving (i).

For (ii), the indicated sublevel set is compact by mass continuity. If it is empty, take $U = \emptyset$. Otherwise, apply (i) with ε sufficiently small and choose a finite cubical neighborhood Y of this set such that

$$\sup_{x \in Y} \mathbf{F}(\widehat{\Phi}(x)) < \omega_q^{d-1}(R, H).$$

The definition of the width implies that $((\widehat{\Phi}|_Y)^* \bar{\lambda})^q = 0$. Taking $U = \text{int}_X Y$ and using (3.2) proves the assertion. $\square$

3.2. The Lusternik-Schnirelmann inequality. We now prove the anisotropic version of [37, Section 3.1]. The restriction estimates allow us to compare a global width with the sum of local widths on disjoint subdomains. The cup-product argument ensures that one member of the global sweepout simultaneously satisfies all the required local energy bounds.

We write $|\Omega|$ for Euclidean volume and call two domains similar if one is obtained from the other by a translation and a positive dilation. For convenience, we set

$$\omega_0^{d-1}(\Omega, H) = \widetilde{\omega}_0^{d-1}(\Omega, H) := 0.$$

Proposition 3.3. *Let H be an autonomous integrand satisfying (H1-H3). Let $\Omega_0, \dots, \Omega_N$ be bounded connected Lipschitz domains in $\mathbb{R}^d$ with unit volume. For each $i = 1, \dots, N$, let $\Omega_i^* \subset \Omega_0$ be similar to Ω_i , and assume that the domains Ω_i^* have pairwise disjoint interiors. We set*

$$v_i := |\Omega_i^*|, \quad V := \min_{1 \leq i \leq N} v_i, \quad p_i := \lfloor p v_i \rfloor.$$

There exists $c = c(\lambda, d, \Omega_0) > 0$ such that

$$\widetilde{\omega}_p^{d-1}(\Omega_0, H) \geq \sum_{i=1}^N v_i \widetilde{\omega}_{p_i}^{d-1}(\Omega_i, H) - \frac{c}{pV} \quad \text{for every } p \geq 1.$$

Proof. We follow the proof of [37, Section 3.1], using Lemma 3.2 in place of their Lemma 2.15. Additivity of the anisotropic energy over disjoint interiors and the same relative cup-product argument give

$$\omega_p^{d-1}(\Omega_0, H) \geq \sum_{i=1}^N \omega_{q_i}^{d-1}(\Omega_i^*, H), \quad q_i \geq 0, \quad \sum_i q_i \leq p. \quad (3.3)$$

Translation invariance and scaling by r^{d-1} give the same normalization as in the cited proof.

Taking $q_i = p_i$, we have $p_i/(pv_i) \geq 1/2$ whenever $p_i \geq 1$. Thus the preceding inequality and Lemma 3.1 imply

$$\sum_{p_i \geq 1} v_i \tilde{\omega}_{p_i}^{d-1}(\Omega_i, H) \leq 2^{1/d} \tilde{\omega}_p^{d-1}(\Omega_0, H) \leq 2^{1/d} C_{\lambda, d, \Omega_0}.$$

The same integer-rounding estimate as in [37] therefore bounds the error by $2^{1/d} C_{\lambda, d, \Omega_0}/(pV)$. With the convention at index zero, this proves the assertion. $\square$

3.3. The frozen constant. We first establish convergence on the unit cube $C = (0, 1)^d$ and then pass to bounded connected Lipschitz domains using the packing arguments of [37]. We also record the orthogonal invariance and Lipschitz continuity of the resulting coefficient.

Proof of Theorem 1.1. Fix $q \geq 1$ and, for $p \geq q$, set $m = \lfloor (p/q)^{1/d} \rfloor$. Partitioning C into m^d equal cubes, the unnormalized Lusternik-Schnirelmann inequality (3.3) and scaling give

$$\tilde{\omega}_p^{d-1}(C, H) \geq \left(\frac{m^d q}{p} \right)^{1/d} \tilde{\omega}_q^{d-1}(C, H).$$

Since $m^d q/p \rightarrow 1$, letting $p \rightarrow \infty$ and then taking the supremum over q yields

$$\liminf_{p \rightarrow \infty} \tilde{\omega}_p^{d-1}(C, H) \geq \sup_{q \geq 1} \tilde{\omega}_q^{d-1}(C, H) \geq \limsup_{p \rightarrow \infty} \tilde{\omega}_p^{d-1}(C, H).$$

This proves (1.3); positivity and finiteness follow from Lemma 3.1.

For unit-volume Lipschitz domains, the packing arguments of [37, Lemmas 3.4-3.6] apply using Proposition 3.3. These constructions require only translations and positive dilations, which preserve the autonomous energy and multiply it by r^{d-1} , respectively. They therefore give

$$\liminf_{p \rightarrow \infty} \tilde{\omega}_p^{d-1}(\Omega, H) \geq a(H) \geq \limsup_{p \rightarrow \infty} \tilde{\omega}_p^{d-1}(\Omega, H).$$

Finally, dilation of a general domain to unit volume gives (1.4). $\square$

The domain independence of the frozen coefficient implies its invariance under orthogonal changes of coordinates.

Corollary 3.4. *Let H satisfy H1-H3. For every $Q \in O(d)$, we have*

$$a(H \circ Q) = a(H), \quad (H \circ Q)(T) := H(QT).$$

Proof. Orthogonal pushforward preserves the sweepout class and gives

$$\tilde{\omega}_p^{d-1}(C, H \circ Q) = \tilde{\omega}_p^{d-1}(Q(C), H).$$

Since $|Q(C)| = 1$, letting $p \rightarrow \infty$ and applying Theorem 1.1 proves the assertion. $\square$

Lemma 3.5. *Let H and K satisfy H1-H3 with the same comparability constant λ . Then*

$$|a(H) - a(K)| \leq a(1) \lambda^{-2} \|H - K\|_\infty.$$

Proof. Since $H, K \geq \lambda$, we have

$$K \leq (1 + \|H - K\|_\infty / \lambda) H, \quad H \leq (1 + \|H - K\|_\infty / \lambda) K.$$

Comparing the energies of the same sweepouts and passing to the Weyl limits gives the corresponding inequalities for $a(H)$ and $a(K)$. Since $a(H), a(K) \leq \lambda^{-1} a(1)$, the conclusion follows. $\square$

Remark 3.6. Consequently, for a continuous variable anisotropy F satisfying [H1-H3](#), the function $x \mapsto a(F(x, \cdot))$ is intrinsically defined and continuous. Indeed, [Corollary 3.4](#) makes its definition independent of the orthonormal frame, and [Lemma 3.5](#) gives continuity in every local orthonormal frame.

4. THE VARIABLE ANISOTROPY WEYL LAW

The main point of departure from the established Weyl law for the area functional in [\[37\]](#) is the case of variables anisotropies. Even on euclidean domains, the self similar strategy breaks down and this is the reason we develop the explicit upper bound constructions in this section. The lower bound follows from the same techniques, using the Lusternick-Schnirelman inequality. For the upper bound we perform a construction which then reduces the problem to an *integer allocation*. This in turn selects the best constants and the integral Weyl law in the limit.

4.1. The lower bound. Here we collect some preliminaries. First we show a readily available improvement of the Lusternik-Schnirelmann inequality for variables anisotropies. This will indeed provide the lower bound for the variable Weyl Law.

Proposition 4.1. *Let (M^d, g) be compact, connected, and smooth, with possibly empty smooth boundary, and let F satisfy [H1-H3](#). Set $C = (0, 1)^d$. Let $C_1, \dots, C_N$ be normal-coordinate cubes with pairwise disjoint interiors and $\overline{C_i} \subset \text{int } M$. Write x_i and r_i for their centers and side lengths, and define the frozen integrands in the center of each cube $H_i = F(x_i, \cdot)$ in the orthonormal coordinates defining C_i . For $p \geq 1$, set:*

- $a_i = a(H_i)$ and $v_i = \text{Vol}_g(C_i)$;
- $S = \sum_{j=1}^N a_j^{d/(d-1)} v_j$;
- $q_i = a_i^{d/(d-1)} v_i / S$ and $p_i = \lfloor pq_i \rfloor$;
- $v_* = \min_i v_i$ and $r_* = \max_i r_i$.

There are $r_0 > 0$, $C_0 = C_0(M, g, d, \lambda)$, and a function $\eta_{F,g}(r) \rightarrow 0$ as $r \downarrow 0$ (depending on the modulus of continuity of F) such that, if $r_* \leq r_0$, then

$$\tilde{\omega}_p^{d-1}(M, F) \geq \frac{S^{(d-1)/d}}{1 + \eta_{F,g}(r_*)} \sum_{i=1}^N q_i \frac{\tilde{\omega}_{p_i}^{d-1}(C, H_i)}{a_i} - \frac{C_0}{pv_*}. \quad (4.1)$$

Proof. We first extend the restriction argument of [Lemma 3.2](#) to an interior cube $R = C_i$ and the variable integrand F . For a relative cycle $\tau = P(T)$ in M , write

$$\mathbf{F}(\tau \llcorner R) := \mathbf{F}(|T|, R).$$

This does not depend on the representative and, by [H1](#), is continuous as a function of τ in the strong mass topology.

The inward deformations used in the proof of [Lemma 3.2](#) can be made in the coordinate chart of R . They converge to the identity in C^1 . Positivity and uniform continuity of F imply that pushing a chain back from an inward deformation of R increases its anisotropic energy by a factor tending to one. The slicing and interpolation argument of that lemma therefore gives, for every $\theta > 0$, a strong mass-continuous restriction $\widehat{\Phi}$ of a strong mass-continuous family $\Phi : X \rightarrow \mathcal{Z}_{d-1, \text{rel}}^0(M, \partial M; \mathbb{Z}_2)$ such that

$$\mathbf{F}(\widehat{\Phi}(x)) \leq (1 + \theta) \mathbf{F}(\Phi(x) \llcorner R) \quad \text{for every } x \in X, \quad \widehat{\Phi}^* \bar{\lambda} = \Phi^* \bar{\lambda}.$$

Here $\widehat{\Phi}$ takes values in the relative cycle space of R . As in [Lemma 3.2\(ii\)](#), it follows that, for $m \geq 1$ and $\delta > 0$, the set

$$\left\{ x \in X : \mathbf{F}(\Phi(x) \llcorner R) \leq \omega_m^{d-1}(R, F) - \delta \right\}$$

is contained in an open set $U \subset X$ on which $(\Phi^* \bar{\lambda}|_U)^m = 0$.

We apply this observation to the cubes C_i . Let $\Phi \in \mathcal{P}_p^{d-1}(M)$, put $X = \text{dmn}(\Phi)$, and write $\alpha = \Phi^* \bar{\lambda}$. We omit indices with $p_i = 0$; if there are no remaining indices, the assertion is immediate. Since $\sum_i q_i = 1$, we have $\sum_i p_i \leq p$. For $\delta > 0$, the preceding restriction argument supplies open sets $U_i \subset X$ containing

$$\left\{ x \in X : \mathbf{F}(\Phi(x) \lfloor C_i) \leq \omega_{p_i}^{d-1}(C_i, F) - \frac{\delta}{N} \right\}$$

and satisfying $(\alpha|_{U_i})^{p_i} = 0$. If the U_i covered X , their relative cup product would give $\alpha^{\sum_i p_i} = 0$, and hence $\alpha^p = 0$. This contradicts the p -sweepout condition. Thus some $x \in X$ lies outside every U_i . Disjointness of the cube interiors gives

$$\mathbf{F}(\Phi(x)) \geq \sum_i \mathbf{F}(\Phi(x) \lfloor C_i) \geq \sum_i \omega_{p_i}^{d-1}(C_i, F) - \delta.$$

Taking the infimum over Φ and letting $\delta \downarrow 0$, we obtain

$$\omega_p^{d-1}(M, F) \geq \sum_i \omega_{p_i}^{d-1}(C_i, F). \quad (4.2)$$

Let $E_i : \mathbb{R}^d \rightarrow T_{x_i} M$ be the orthonormal identification defining C_i , and put

$$\Psi_i(z) = \exp_{x_i}(r_i E_i(z - (1/2, \dots, 1/2))), \quad z \in C.$$

We compare each local width with its frozen counterpart. For $z \in C$ and $T \in G_{d-1}(\mathbb{R}^d)$, the integrand pulled back by Ψ_i , after removing the dilation factor, is

$$H_{i,r_i}(z, T) := r_i^{-(d-1)} J_{d-1} \Psi_i(z, T) F(\Psi_i(z), D\Psi_i(z)T).$$

Normal-coordinate estimates and continuity of F on the compact Grassmannian bundle give a function $\rho_{F,g}(r) \rightarrow 0$ such that

$$\sup_{\substack{z \in C \\ T \in G_{d-1}(\mathbb{R}^d)}} |H_{i,r_i}(z, T) - H_i(T)| \leq \rho_{F,g}(r_*) \quad \text{for every } i.$$

The function can be chosen uniformly over all admissible normal-coordinate cubes of side at most r_* . Since $H_i \geq \lambda$ and $v_i = r_i^d(1 + O_g(r_i^2))$, comparison under Ψ_i gives, after decreasing r_0 if necessary,

$$\omega_m^{d-1}(C_i, F) \geq \frac{v_i^{(d-1)/d}}{1 + \eta_{F,g}(r_*)} \omega_m^{d-1}(C, H_i) \quad (m \geq 1), \quad (4.3)$$

where one may take $\eta_{F,g}(r) = C_{M,g,\lambda}(r^2 + \rho_{F,g}(r))$. For $F \in C^1$, the normal-coordinate estimates instead give $\rho_{F,g}(r) \leq C_{M,g,\|F\|_{C^1}} r$.

Set $\beta = (d-1)/d$ and, for $p_i \geq 1$, write $b_i = \tilde{\omega}_{p_i}^{d-1}(C, H_i)$. Equations (4.2) and (4.3) imply

$$\tilde{\omega}_p^{d-1}(M, F) \geq \frac{1}{1 + \eta_{F,g}(r_*)} \sum_{p_i \geq 1} v_i^\beta \left(\frac{p_i}{p} \right)^{1/d} b_i.$$

For each index in this sum, let $t_i = p_i/(pq_i)$. Then $1/2 \leq t_i \leq 1$ and

$$0 \leq 1 - t_i^{1/d} \leq 1 - t_i < \frac{1}{pq_i}.$$

The definitions of S and q_i give the identity

$$v_i^\beta q_i^{1/d} = S^\beta \frac{q_i}{a_i}.$$

Consequently, replacing $t_i^{1/d}$ by 1 in the preceding lower bound incurs an error of at most

$$\frac{S^\beta}{p} \sum_{p_i \geq 1} \frac{b_i}{a_i}.$$

By Lemma 3.1, the b_i are uniformly bounded in terms of d and λ . Moreover, $a_i \geq \lambda a(1)$, $S \leq C_{d,\lambda} \text{Vol}_g(M)$, and $Nv_* \leq \sum_i v_i \leq \text{Vol}_g(M)$. The error is therefore at most $C/(pv_*)$. Using $\tilde{\omega}_0^{d-1}(C, H_i) = 0$ to restore the omitted indices proves (4.1). $\square$

The immediate corollary of Proposition 4.1 is the lower bound for the Weyl law:

Corollary 4.2. *Let (M^d, g) and F satisfy H1-H3. Then*

$$\liminf_{p \rightarrow \infty} \tilde{\omega}_p^{d-1}(M, F) \geq \left(\int_M a(F(x, \cdot))^{d/(d-1)} dV_g(x) \right)^{(d-1)/d}.$$

Proof. Fix a finite packing as in Proposition 4.1. For this packing, $p_i \rightarrow \infty$ as $p \rightarrow \infty$. The frozen Weyl law and (4.1) give

$$\liminf_{p \rightarrow \infty} \tilde{\omega}_p^{d-1}(M, F) \geq \frac{S^{(d-1)/d}}{1 + \eta_{F,g}(r_*)} \sum_{i=1}^N q_i \lim_{p \rightarrow \infty} \frac{\tilde{\omega}_{p_i}^{d-1}(C, H_i)}{a_i} = \frac{S^{(d-1)/d}}{1 + \eta_{F,g}(r_*)}.$$

Now choose packings whose maximal side length tends to zero and whose unions exhaust M up to vanishing volume. By continuity of $x \mapsto a(F(x, \cdot))$,

$$S = \sum_{i=1}^N \int_{C_i} a(F(x_i, \cdot))^{d/(d-1)} dV_g(x) \longrightarrow \int_M a(F(x, \cdot))^{d/(d-1)} dV_g(x).$$

Finally, (again by continuity) $\eta_{F,g}(r_*) \rightarrow 0$, which proves the claim. $\square$

4.2. An equivariant cube to slab map. For ease of notation we name the collection of sets of finite perimeter

$$\mathcal{C}(M) := \mathbf{I}_d(M; \mathbb{Z}_2).$$

In the sequel we will need to repeatedly use a map that takes a sweepout (of sets of finite perimeter) in a cube and with a controlled amount of cuts and translations, fits it into a thin slab with equal volume. Indeed we cannot use any rotation or direct anisotropic scaling, since these type of operations will change the anisotropic mass. We also need this map to be equivariant, meaning that it should commute with set complementing. The interested reader may skip the proof at a first reading and only inspect the figures 1 to 4. We describe this map in detail in the following proposition:

Proposition 4.3. *There exists a family of maps*

$$T_\nu : \mathcal{C}([0, 1]^d) \longrightarrow \mathcal{C}([0, \nu] \times [0, 1]^{d-1}), \quad \nu \in [0, 1],$$

with the following properties:

(1) *Let*

$$\iota_\nu : \mathcal{C}([0, \nu] \times [0, 1]^{d-1}) \longrightarrow \mathcal{C}([0, 1]^d)$$

be the map induced by set inclusion. Then $(\nu, S) \mapsto \iota_\nu(T_\nu(S))$ is jointly continuous in the usual topology on $[0, 1]$ and the flat topology on fillings.

(2) *The map is equivariant: for every $S \in \mathcal{C}([0, 1]^d)$,*

$$T_\nu([0, 1]^d \setminus S) = ([0, \nu] \times [0, 1]^{d-1}) \setminus T_\nu(S).$$

(3) *For every fixed autonomous anisotropy H satisfying H1-H3 and every $S \in \mathcal{C}([0, 1]^d)$,*

$$\mathbf{M}_H(\partial T_\nu(S)) \leq \nu^{\frac{d-1}{d}} \mathbf{M}_H(\partial S) + C(d, \lambda),$$

where $C(d, \lambda) > 0$ depends only on d and λ .

(4) The maps preserve no concentration of mass uniformly in ν . More precisely, if a family $\mathcal{E} \subset \mathcal{C}([0, 1]^d)$ satisfies

$$\lim_{r \downarrow 0} \sup_{\substack{S \in \mathcal{E} \\ y \in \mathbb{R}^d}} \|\partial S\|(B_r(y) \cap (0, 1)^d) = 0,$$

then

$$\lim_{r \downarrow 0} \sup_{\substack{\nu \in [0, 1] \\ S \in \mathcal{E} \\ y \in \mathbb{R}^d}} \|\partial T_\nu(S)\|(B_r(y)) = 0.$$

Proof. Given $\rho_1, \rho_2 > 0$ and $0 < \theta < 1$, we first show how to rearrange the rectangle $[0, \rho_1] \times [0, \rho_2]$ into $[0, \theta\rho_1] \times [0, \rho_2/\theta]$ in four steps. We disregard the cutting lines and identify sets modulo null sets throughout.

For $(x, y) \in [0, \rho_1] \times [0, \rho_2]$, define (see Figure 1)

$$\tau_{\rho_1, \rho_2; \theta}^{(1)}(x, y) = \begin{cases} (x, y), & \rho_2 x + (1 - \theta)\rho_1 y < \rho_1 \rho_2, \\ (x - \rho_1, y), & \rho_2 x + (1 - \theta)\rho_1 y > \rho_1 \rho_2. \end{cases}$$

The cut joins $(\theta\rho_1, \rho_2)$ to $(\rho_1, 0)$. The upper-right triangle is translated by $(-\rho_1, 0)$, while the remaining piece is fixed. The resulting parallelogram has vertices

$$(0, 0), \quad (\rho_1, 0), \quad (\theta\rho_1, \rho_2), \quad (-(1 - \theta)\rho_1, \rho_2).$$

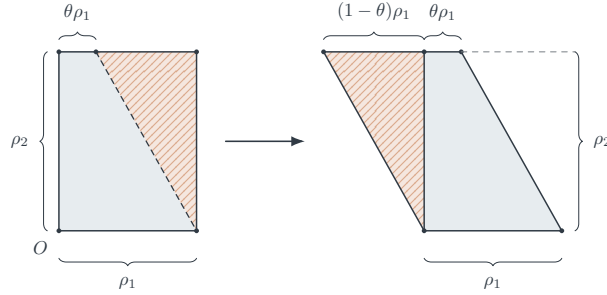


FIGURE 1. The cut and translation of the map $\tau_{\rho_1, \rho_2; \theta}^{(1)}$.

On the parallelogram obtained in the first step, define (see Figure 2)

$$\tau_{\rho_1, \rho_2; \theta}^{(2)}(x, y) = \begin{cases} (x, y), & \rho_2 x < \theta\rho_1 y, \\ (x - (1 - \theta)\rho_1, y + \rho_2), & \rho_2 x > \theta\rho_1 y. \end{cases}$$

The cut joins $(0, 0)$ to $(\theta\rho_1, \rho_2)$. The triangle on its right is translated by $(-(1 - \theta)\rho_1, \rho_2)$, while the other triangle is fixed. The resulting parallelogram has vertices

$$(0, 0), \quad (\theta\rho_1, \rho_2), \quad ((2\theta - 1)\rho_1, 2\rho_2), \quad (-(1 - \theta)\rho_1, \rho_2).$$

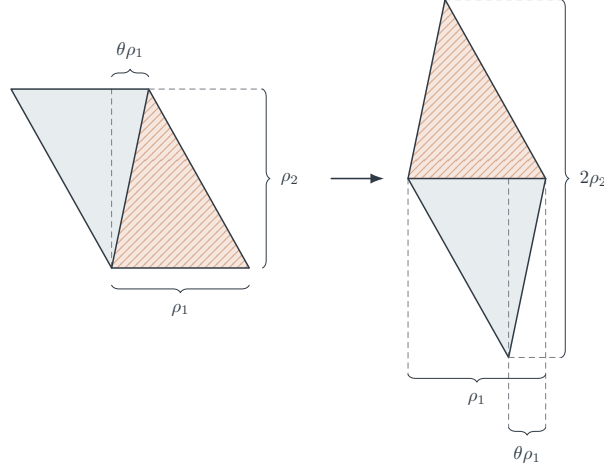


FIGURE 2. The cut and translation of the map $\tau_{\rho_1, \rho_2; \theta}^{(2)}$.

On the parallelogram obtained in the second step, define (see Figure 3)

$$\tau_{\rho_1, \rho_2; \theta}^{(3)}(x, y) = \left(x - \theta \rho_1 \left\lfloor \frac{x}{\theta \rho_1} \right\rfloor, \quad y - \rho_2 \left\lfloor \frac{x}{\theta \rho_1} \right\rfloor \right).$$

Away from the vertical cutting lines, the integer

$$- \left\lfloor \frac{x}{\theta \rho_1} \right\rfloor$$

counts the strips from right to left, starting at zero. Thus the rightmost strip is fixed; each successive strip is translated one additional distance $\theta \rho_1$ to the right and ρ_2 upward. The possibly narrower leftmost piece is included in the same formula.

The resulting parallelogram is

$$\left\{ (x, y) : 0 \leq x \leq \theta \rho_1, \quad \frac{\rho_2 x}{\theta \rho_1} \leq y \leq \frac{\rho_2 x}{\theta \rho_1} + \frac{\rho_2}{\theta} \right\}.$$

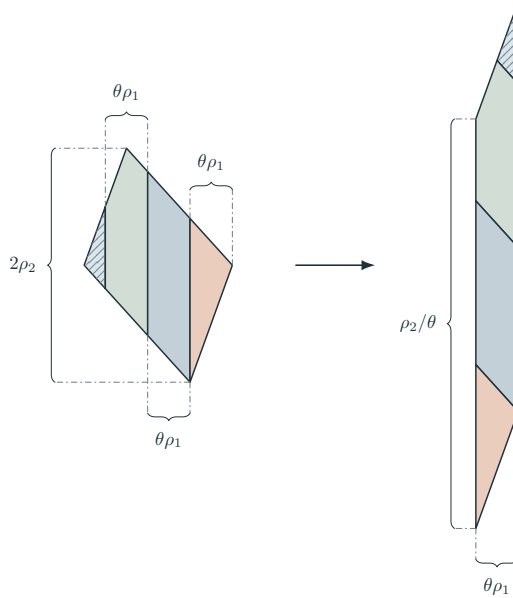


FIGURE 3. The multiple cuts and translations of $\tau_{\rho_1, \rho_2; \theta}^{(3)}$.

On the parallelogram obtained in the third step, define (see Figure 4)

$$\tau_{\rho_1, \rho_2; \theta}^{(4)}(x, y) = \begin{cases} (x, y), & y < \rho_2/\theta, \\ (x, y - \rho_2/\theta), & y > \rho_2/\theta. \end{cases}$$

The horizontal cut is $y = \rho_2/\theta$. The triangular cap above this line is translated downward by $(0, -\rho_2/\theta)$, filling the triangular gap at the bottom. The resulting rectangle is

$$[0, \theta\rho_1] \times \left[0, \frac{\rho_2}{\theta}\right].$$

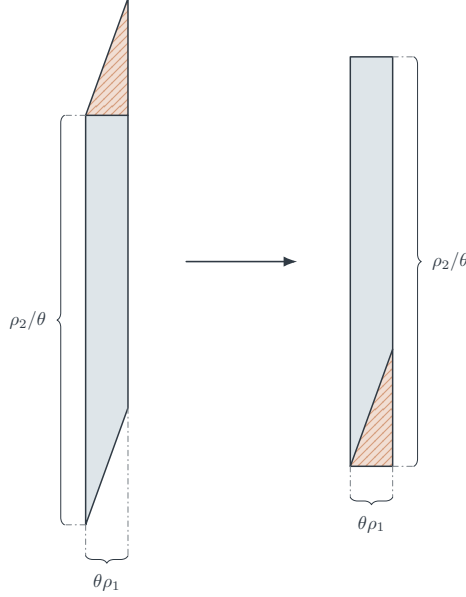


FIGURE 4. The final cut and translation of $\tau_{\rho_1, \rho_2; \theta}^{(4)}$ turns the parallelogram into a rectangle.

The resulting planar map

$$\mathcal{T}_{\rho_1, \rho_2; \theta} : [0, \rho_1] \times [0, \rho_2] \longrightarrow [0, \theta\rho_1] \times [0, \rho_2/\theta]$$

is defined almost everywhere by

$$\mathcal{T}_{\rho_1, \rho_2; \theta} := \tau_{\rho_1, \rho_2; \theta}^{(4)} \circ \tau_{\rho_1, \rho_2; \theta}^{(3)} \circ \tau_{\rho_1, \rho_2; \theta}^{(2)} \circ \tau_{\rho_1, \rho_2; \theta}^{(1)}.$$

We now define T_ν . For $0 < \nu < 1$, set

$$\rho := \nu^{1/d}, \quad \phi_\nu : [0, 1]^d \longrightarrow [0, \rho]^d, \quad \phi_\nu(x) := \rho x.$$

The first rearrangement is

$$T_\nu^1(x_1, \dots, x_d) := ([\mathcal{T}_{\rho, \rho; \rho}(x_1, x_2)]_1, [\mathcal{T}_{\rho, \rho; \rho}(x_1, x_2)]_2, x_3, \dots, x_d),$$

where $[\cdot]_1$ and $[\cdot]_2$ denote the two coordinate components. Thus $T_\nu^1 \circ \phi_\nu$ sends $[0, 1]^d$ onto $[0, \rho^2] \times [0, 1] \times [0, \rho]^{d-2}$ modulo null sets. When $d \geq 3$, the second rearrangement is

$$T_\nu^2(x_1, \dots, x_d) := ([\mathcal{T}_{\rho^2, \rho; \rho}(x_1, x_3)]_1, x_2, [\mathcal{T}_{\rho^2, \rho; \rho}(x_1, x_3)]_2, x_4, \dots, x_d).$$

The composition $T_\nu^2 \circ T_\nu^1 \circ \phi_\nu$ sends $[0, 1]^d$ onto $[0, \rho^3] \times [0, 1]^2 \times [0, \rho]^{d-3}$ modulo null sets. In general, for $k = 1, \dots, d-1$, define

$$T_\nu^k(x_1, \dots, x_d) := ([\mathcal{T}_{\rho^k, \rho; \rho}(x_1, x_{k+1})]_1, x_2, \dots, x_k, [\mathcal{T}_{\rho^k, \rho; \rho}(x_1, x_{k+1})]_2, x_{k+2}, \dots, x_d),$$

with empty coordinate lists omitted. At step k , the side length in the e_1 direction is multiplied by ρ , and the side length in the e_{k+1} direction is multiplied by ρ^{-1} . The successive boxes are therefore

$$\begin{aligned} [0, 1]^d &\xrightarrow{\phi_\nu} [0, \rho]^d \xrightarrow{T_\nu^1} [0, \rho^2] \times [0, 1] \times [0, \rho]^{d-2} \\ &\xrightarrow{T_\nu^2} \dots \xrightarrow{T_\nu^k} [0, \rho^{k+1}] \times [0, 1]^k \times [0, \rho]^{d-k-1} \\ &\xrightarrow{T_\nu^{k+1}} \dots \xrightarrow{T_\nu^{d-1}} [0, \rho^d] \times [0, 1]^{d-1} = [0, \nu] \times [0, 1]^{d-1}. \end{aligned}$$

Define the map:

$$F_\nu := T_\nu^{d-1} \circ \dots \circ T_\nu^1 \circ \phi_\nu$$

be the underlying point map. Then for any set of finite perimeter $S \in \mathcal{C}([0, 1]^d)$, define

$$T_\nu(S) := (F_\nu)_\# S,$$

where the image is taken piecewise on the domains of the defining formulas, modulo null sets. At the endpoints, set $T_0(S) := \emptyset$, $T_1(S) := S$, and $F_1 := \text{id}_{[0, 1]^d}$. The equivariance follows immediately since the operation is a set map.

We prove the joint continuity of $\iota_\nu \circ T_\nu$. Throughout, we identify sets modulo null sets and regard $T_\nu(S)$ as a subset of $[0, 1]^d$ via ι_ν . For these top-dimensional fillings, flat convergence agrees with convergence of characteristic functions in L^1 .

Since the rearrangements are translations on finitely many pieces and are bijective almost everywhere, for $0 < \nu < 1$ we have

$$|T_\nu(S)| = \nu|S|, \quad |T_\nu(S) \triangle T_\nu(S')| = \nu|S \triangle S'|,$$

where $|\cdot|$ denotes Lebesgue measure.

Fix $\nu_0 \in (0, 1]$. Locally near ν_0 , the number of pieces is uniformly bounded. Away from the finitely many limiting cutting hyperplanes, the branch inequalities and the integers in the floor functions stabilize, while the corresponding translations vary continuously. At $\nu_0 = 1$, the limiting second and fourth translations cancel, and the composite is the identity. Consequently,

$$F_\nu(x) \longrightarrow F_{\nu_0}(x) \quad \text{for almost every } x \in [0, 1]^d.$$

For fixed S and every continuous function h on $[0, 1]^d$, the change-of-variables formula and dominated convergence give

$$\int_{T_\nu(S)} h(y) dy = \nu \int_S h(F_\nu(x)) dx \longrightarrow \nu_0 \int_S h(F_{\nu_0}(x)) dx = \int_{T_{\nu_0}(S)} h(y) dy.$$

Since the characteristic functions are bounded by one, approximating $\mathbf{1}_{T_{\nu_0}(S)}$ in L^1 by continuous functions shows that $|T_\nu(S) \cap T_{\nu_0}(S)| \rightarrow |T_{\nu_0}(S)|$. Together with $|T_\nu(S)| = \nu|S|$, this proves $|T_\nu(S) \triangle T_{\nu_0}(S)| \rightarrow 0$. At $\nu_0 = 0$, continuity follows uniformly in S from $|T_\nu(S)| \leq \nu$. Finally,

$$|T_\nu(S) \triangle T_{\nu_0}(S_0)| \leq \nu|S \triangle S_0| + |T_\nu(S_0) \triangle T_{\nu_0}(S_0)|$$

proves joint continuity.

It remains to prove the estimate in point (3). Write

$$P_H(E; U) := \mathbf{M}_H(\partial E \llcorner U).$$

Since H is autonomous, each translation preserves the anisotropic perimeter inside the translated piece. Moreover in the whole operations, we are never using any rotation, hence every perimeter is transferred identically, and some cut pieces will be added which we will subsequently estimate.

By the BV gluing formula and the comparability assumption, a new gluing interface contributes at most $C(\lambda)$ times its $(d-1)$ -dimensional measure.

In each planar operation, the newly cut edges become exterior edges of the intermediate target domain. The total length of the new interior gluing interfaces over the four operations is bounded by $C(\rho_1 + \rho_2)$, uniformly in θ . Indeed, the first two interfaces have lengths ρ_2 and ρ_1 , respectively. In the third operation, the vertical cuts become exterior edges, while the gluing interfaces are paired translated portions of the original slanted sides. The fourth operation glues the two remaining slanted sides. Their lengths are bounded by $C(\rho_1 + \rho_2)$.

Recall that $\rho = \nu^{1/d}$. At step $k = 1, \dots, d-1$, we use $\rho_1 = \rho^k$, $\rho_2 = \rho$, and $\theta = \rho$. The untouched coordinates have total transverse volume ρ^{d-k-1} , so this step increases the relative perimeter by at most

$$C(\lambda)(\rho^k + \rho)\rho^{d-k-1}.$$

The initial homothety multiplies anisotropic perimeter by ρ^{d-1} . Summing over the rearrangements therefore gives

$$\begin{aligned} P_H(T_\nu(S); (0, \nu) \times (0, 1)^{d-1}) &\leq \rho^{d-1} P_H(S; (0, 1)^d) + C(\lambda) \sum_{k=1}^{d-1} (\rho^k + \rho) \rho^{d-k-1} \\ &\leq \nu^{\frac{d-1}{d}} P_H(S; (0, 1)^d) + C(d, \lambda) \nu^{1/d}. \end{aligned}$$

Finally, the boundary of the target rectangle has anisotropic mass bounded by $C(d, \lambda)$, uniformly in ν . Adding its possible contribution yields

$$\mathbf{M}_H(\partial T_\nu(S)) \leq \nu^{\frac{d-1}{d}} \mathbf{M}_H(\partial S) + C(d, \lambda).$$

The endpoint cases follow from $T_0(S) = \emptyset$ and $T_1(S) = S$.

We finally prove (4). The hypothesis and a finite covering of $[0, 1]^d$ by small balls imply

$$\sup_{S \in \mathcal{E}} \|\partial S\|((0, 1)^d) < \infty.$$

Fix $\delta \in (0, 1)$. For $\nu \in [\delta, 1]$, the number of pieces in the construction is uniformly bounded. On each piece, the underlying map is a homothety of factor $\nu^{1/d}$ followed by a translation. All additional boundary contributions lie in a uniformly bounded number of affine hyperplanes. Hence

$$\sup_{\substack{\nu \in [\delta, 1] \\ S \in \mathcal{E} \\ y \in \mathbb{R}^d}} \|\partial T_\nu(S)\|(B_r(y)) \leq C(d, \delta) \sup_{\substack{S \in \mathcal{E} \\ y \in \mathbb{R}^d}} \|\partial S\|(B_{r\delta^{-1/d}}(y) \cap (0, 1)^d) + C(d, \delta) r^{d-1}.$$

For fixed δ , this tends to zero as $r \downarrow 0$.

For $0 < \nu \leq \delta$, we use the relative perimeter estimate proved above with $H = 1$. The boundary of the target slab has $2d$ faces, each contributing at most $C(d)r^{d-1}$ inside any ball of radius r . Therefore

$$\sup_{\substack{\nu \in [0, \delta] \\ S \in \mathcal{E} \\ y \in \mathbb{R}^d}} \|\partial T_\nu(S)\|(B_r(y)) \leq \delta^{(d-1)/d} \sup_{S \in \mathcal{E}} \|\partial S\|((0, 1)^d) + C(d)\delta^{1/d} + C(d)r^{d-1},$$

where $\nu = 0$ contributes nothing. Combining the two estimates, first letting $r \downarrow 0$ and then $\delta \downarrow 0$, proves (4). $\square$

Remark 4.4. There are many ways to construct the map in Proposition 4.3 and the construction already done above has the advantage that the multiplier behind the perimeter in the estimate, is indeed one which makes subsequent computations in Lemma 4.6 more intuitive to follow. One can imagine to cut the square after appropriate rescaling into a series of smaller ν rectangles with some leftovers and then place a scaling of the whole sweepout in the left over that is not covered in the target. This would indeed introduce a multiplicative error $(1 + o(1))$ depending on the rationality of some power of ν which then makes the computations later much harder to follow. We chose to present the preceeding

construction, since the heart of the proof which is the *integer allocation* estimate in Lemma 4.6 would be better presented.

4.3. Competitor sweepouts In a long rectangle. In this section we prove the upper bound using a variational construction. We first recall the definition of the join of two topological spaces X, Y :

$$X * Y := \frac{X \times Y \times [0, 1]}{\sim},$$

where the equivalence relation is generated by

$$\begin{cases} (x, y_1, 0) \sim (x, y_2, 0) & \text{for all } x \in X, y_1, y_2 \in Y, \\ (x_1, y, 1) \sim (x_2, y, 1) & \text{for all } x_1, x_2 \in X, y \in Y. \end{cases}$$

The use of joins of filling double covers to construct higher sweepouts is also present in [23, Section 3]. Now for $N \in \mathbb{N}$, consider the rectangle

$$Q = [0, N] \times [0, 1]^{d-1} = \bigcup_{i=1}^N C_i,$$

where $\{C_i\}_{i=1}^N$ are unit cubes, equipped respectively with frozen integrands H_i satisfying H1 and H3. Let

$$\Phi_i : X_i \longrightarrow \mathcal{Z}_{d-1, \text{rel}}^0(C_i, \partial C_i; \mathbb{Z}_2), \quad i = 1, \dots, N,$$

be p -sweepouts which are almost optimal for H_i :

$$\sup_{x \in X_i} \mathbf{M}_{H_i}(\Phi_i(x)) \leq \omega_p^{d-1}(C_i, H_i) + \varepsilon. \quad (4.4)$$

To work with sets of finite perimeter and set operations, we first pass to the pullback double covers for all $i = 1, \dots, N$

$$\tilde{X}_i := \{(x, E) \in X_i \times \mathcal{C}(C_i) : \partial_{\text{rel}}[E] = \Phi_i(x)\},$$

where $\mathcal{C}(C_i)$ denotes the space of sets of finite perimeter in C_i , identified up to null sets. The covering projection is $\pi_i(x, E) = x$, and its deck involution is

$$\tau_i(x, E) = (x, C_i \setminus E).$$

The lifted filling maps

$$\tilde{\Phi}_i : \tilde{X}_i \longrightarrow \mathcal{C}(C_i), \quad \tilde{\Phi}_i(x, E) = E,$$

satisfy

$$\partial_{\text{rel}}[\tilde{\Phi}_i(\tilde{x})] = \Phi_i(\pi_i(\tilde{x})), \quad \tilde{\Phi}_i(\tau_i \tilde{x}) = C_i \setminus \tilde{\Phi}_i(\tilde{x}).$$

We equip produce space $\tilde{X}_1 \times \dots \times \tilde{X}_N$ with the diagonal involution

$$(\tilde{x}_1, \dots, \tilde{x}_N) \mapsto (\tau_1 \tilde{x}_1, \dots, \tau_N \tilde{x}_N)$$

and form the join

$$\tilde{Z}_K := (\tilde{X}_1 \times \dots \times \tilde{X}_N)^{*K} = \underbrace{(\tilde{X}_1 \times \dots \times \tilde{X}_N) * \dots * (\tilde{X}_1 \times \dots \times \tilde{X}_N)}_{K \text{ times}}.$$

A point of $\tilde{Z}_K$ can be formally represented by

$$z = \begin{bmatrix} (t_1; \tilde{x}_1^1, \dots, \tilde{x}_N^1) \\ (t_2; \tilde{x}_1^2, \dots, \tilde{x}_N^2) \\ \dots \\ (t_K; \tilde{x}_1^K, \dots, \tilde{x}_N^K) \end{bmatrix}, \quad t_\ell \geq 0, \quad \sum_{\ell=1}^K t_\ell = N,$$

with $\tilde{x}_i^\ell \in \tilde{X}_i$. Whenever $t_\ell = 0$, the tuple $(\tilde{x}_1^\ell, \dots, \tilde{x}_N^\ell)$ is forgotten. The induced free involution on the join is

$$\tau^{(K)} z = \begin{bmatrix} (t_1; \tau_1 \tilde{x}_1^1, \dots, \tau_N \tilde{x}_N^1) \\ (t_2; \tau_1 \tilde{x}_1^2, \dots, \tau_N \tilde{x}_N^2) \\ \dots \\ (t_K; \tau_1 \tilde{x}_1^K, \dots, \tau_N \tilde{x}_N^K) \end{bmatrix}.$$

The parameter space for the resulting family of relative cycles will be $Z_K := \tilde{Z}_K / \langle \tau^{(K)} \rangle$.

For $z \in \tilde{Z}_K$, define the incremental slabs

$$S_\ell(z) := [s_{\ell-1}, s_\ell] \times [0, 1]^{d-1}, \quad s_0 = 0, \quad s_\ell = \sum_{j=1}^{\ell} t_j, \quad \ell = 1, \dots, K.$$

All partitions are understood up to null sets. If $t_\ell = 0$, then S_ℓ is empty, consistently with the identifications in the join. We further set (see figure 5)

$$S_\ell^i(z) := S_\ell(z) \cap C_i, \quad i = 1, \dots, N.$$

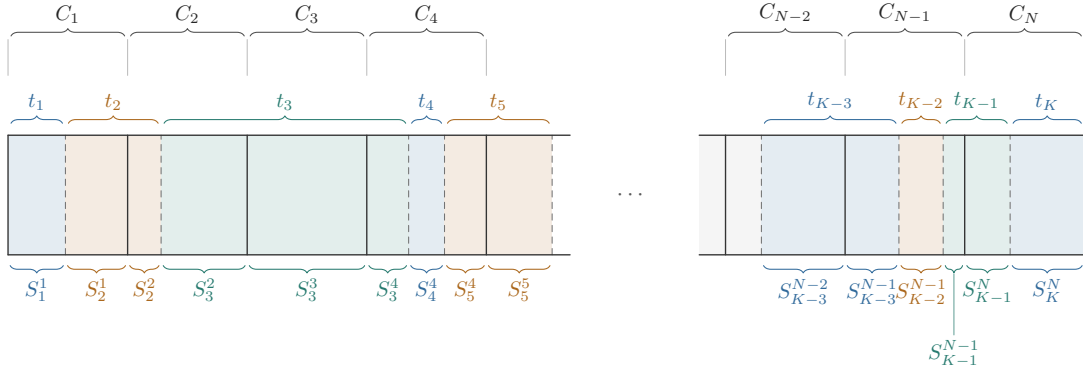


FIGURE 5. Subdivision of the cells into the pieces S_ℓ^i . The dependence on z is suppressed for simplicity.

Now on each nonempty slab S_ℓ^i , we use the maps of Proposition 4.3 and assign sets. First consider ϕ_η to be the set map of shifting in the e_1 direction by η . Further consider the map:

$$T_\ell^i := \phi_{\max\{i-1, s_{\ell-1}\}} \circ T_{|S_\ell^i(z)|} \circ \phi_{1-i} : \mathcal{C}(C_i) \rightarrow \mathcal{C}(S_\ell^i).$$

Here T_ν is the map defined in Proposition 4.3 and $|S_\ell^i(z)|$ denotes its volume, equivalently its width in the first coordinate (here empty slabs contribute the empty set). Then assign the set:

$$E_\ell^i(z) := T_\ell^i(\tilde{\Phi}_i(\tilde{x}_i^\ell)),$$

Pasting these sets of finite perimeter together gives a set $\tilde{\Psi}(z) \in \mathcal{C}(Q)$, precisely defined as:

$$\tilde{\Psi}(z) := \sum_{i=1}^N \sum_{\ell=1}^K E_\ell^i(z).$$

By the equivariance in Proposition 4.3(2) The construction respects complementation:

$$\begin{aligned} \tilde{\Psi}(\tau^{(K)} z) &= \sum_{i=1}^N \sum_{\ell=1}^K T_\ell^i(\tilde{\Phi}_i(\tau_i \tilde{x}_i^\ell)) = \sum_{i=1}^N \sum_{\ell=1}^K T_\ell^i(C_i \setminus \tilde{\Phi}_i(\tilde{x}_i^\ell)) \\ &= \sum_{i=1}^N \sum_{\ell=1}^K (S_\ell^i \setminus E_\ell^i(z)) = Q \setminus \tilde{\Psi}(z), \end{aligned}$$

where in the second-to-last equality we used Proposition 4.3(2). Consequently, its relative boundary defines a map on the quotient,

$$\Psi : Z_K \longrightarrow \mathcal{Z}_{d-1, \text{rel}}^0(Q, \partial Q; \mathbb{Z}_2), \quad \Psi([z]) := \partial_{\text{rel}}[\tilde{\Psi}(z)].$$

Lemma 4.5. *The map Ψ constructed above is indeed a $K(p+1) - 1$ sweepout, continuous in flat topology with no concentration of mass.*

Proof. By Proposition 4.3(1), continuity of translations in L^1 , and continuity of the lifted filling maps, each $E_\ell^i(z)$ depends continuously on the parameters whenever $S_\ell^i(z)$ has positive volume. Continuity also holds when this slab collapses, since

$$|E_\ell^i(z)| \leq |S_\ell^i(z)| \longrightarrow 0.$$

Moreover, as $t_\ell \rightarrow 0$,

$$\sum_{i=1}^N |E_\ell^i(z)| \leq \sum_{i=1}^N |S_\ell^i(z)| = t_\ell \longrightarrow 0,$$

uniformly in the coordinates forgotten by the join. Consequently, $\tilde{\Psi}$ is well defined and continuous in L^1 on $\tilde{Z}_K$. The estimate

$$\mathcal{F}(\partial_{\text{rel}}[E], \partial_{\text{rel}}[E']) \leq |E \Delta E'|$$

and the equivariance of $\tilde{\Psi}$ then show that Ψ is well defined and flat continuous on Z_K .

The family Ψ also has no concentration of mass. Indeed, each Φ_i is mass continuous and therefore has no concentration of mass by Proposition 2.2. Applying Proposition 4.3(4) to the translated lifted fillings $\phi_{1-i}(\tilde{\Phi}_i(\tilde{x}_i))$, and using translation invariance of the supremum over ball centers, gives

$$\lim_{r \downarrow 0} \sup_{\substack{z \in \tilde{Z}_K \\ y \in \mathbb{R}^d}} \|\partial E_\ell^i(z)\|(B_r(y)) = 0 \quad \text{for every } i, \ell.$$

Since the pieces have disjoint interiors, subadditivity of boundary mass yields

$$\sup_{\substack{[z] \in Z_K \\ y \in Q}} \|\Psi([z])\|(B_r(y)) \leq \sum_{i=1}^N \sum_{\ell=1}^K \sup_{\substack{z \in \tilde{Z}_K \\ y \in \mathbb{R}^d}} \|\partial E_\ell^i(z)\|(B_r(y)) \longrightarrow 0.$$

Here the full boundaries of the pieces already include all contributions along the gluing interfaces.

We next verify the sweepout condition. All cohomology groups below have coefficients in $\mathbb{Z}_2$. The characteristic class of the filling double cover is $\bar{\lambda}$: a loop has nontrivial monodromy precisely when its lifted fillings differ at the endpoints by the whole domain, equivalently when its Almgren class is the relative fundamental class. This identification follows from Almgren's isomorphism [3]; see [37, Section 2.5] for the relative formulation and [39, Section 5] for the filling double-cover description in codimension one. Thus, writing w_1 for the first Stiefel-Whitney class of a double cover, we have

$$w_1(\pi_i)^p = (\Phi_i^* \bar{\lambda})^p \neq 0 \quad \text{for every } i.$$

For a finite polyhedron with a free $\mathbb{Z}_2$ -action, its mod-2 cohomological height is the largest exponent for which the corresponding power of w_1 is nonzero. The product formula in [15, Corollary 2.5] states that the height of a diagonal product is the minimum of the heights of its factors. Hence $\tilde{X}_1 \times \cdots \times \tilde{X}_N$ has height at least p .

The join formula in [15, Theorem 2.4] states that the height of a join is the sum of the heights of its factors plus one. This additivity already appears, in the projective-space formulation, in [26]; see also [27, Section 3 and Appendix B]. Iterating the join formula gives

$$w_1(\tilde{Z}_K \longrightarrow Z_K)^{K(p+1)-1} \neq 0.$$

Finally, the equivariant filling map $\tilde{\Psi}$ identifies $\tilde{Z}_K \rightarrow Z_K$ with the pullback under Ψ of the filling double cover of the relative cycle space of Q . Naturality therefore gives

$$\Psi^* \bar{\lambda} = w_1(\tilde{Z}_K \rightarrow Z_K),$$

and consequently

$$\Psi^*(\bar{\lambda}^{K(p+1)-1}) = w_1(\tilde{Z}_K \rightarrow Z_K)^{K(p+1)-1} \neq 0.$$

Since Z_K is a finite polyhedron, it admits a finite cubical structure, so this is a $K(p+1) - 1$ sweepout in the sense of our definition. $\square$

Define the integrand H as follows:

$$H|_{\text{int } C_i} = H_i.$$

On cube faces we set $H = 1$; their contributions will be bounded separately. With the definition above we state the following lemma:

Lemma 4.6. *The following estimate is true:*

$$\begin{aligned} \sup_{x \in \text{dmn}(\Psi)} \mathbf{M}_H(\Psi(x)) &\leq p^{1/d} (K + N - 1)^{\frac{1}{d}} \left(\sum_{i=1}^N a(H_i)^{\frac{d}{d-1}} \right)^{\frac{d-1}{d}} \\ &\quad + o(p^{1/d}) + C(\lambda, N, d)K + C(N, K)\varepsilon. \end{aligned}$$

Proof. First note that, bounding the boundaries of the cubes with some constant $C(N, d, \lambda)$ we get the estimate:

$$\sup_{x \in \text{dmn}(\Psi)} \mathbf{M}_H(\Psi(x)) \leq C(N, d, \lambda) + \sup_{x \in \text{dmn}(\Psi)} \sum_{i=1}^N \sum_{\ell=1}^K \mathbf{M}_{H_i}(\partial E_\ell^i(x)).$$

Here the full boundaries of the pieces include all interfaces between adjacent slabs.

Consider the width of each $S_\ell^i(x)$ to be:

$$w_\ell^i(x) = |S_\ell^i(x)|, \quad \sum_{i=1}^N \sum_{\ell=1}^K w_\ell^i = N.$$

Then we have, by the relative-perimeter estimate proved in Proposition 4.3, adding the boundary of each target slab, and almost optimality of the sweepouts Φ_i in (4.4):

$$\sum_{i=1}^N \sum_{\ell=1}^K \mathbf{M}_{H_i}(\partial E_\ell^i(x)) \leq \sum_{i=1}^N \sum_{\ell=1}^K (w_\ell^i)^{\frac{d-1}{d}} \omega_p^{d-1}(C_i, H_i) + C(\lambda, d)NK + NK\varepsilon.$$

Empty slabs contribute zero. Now for p large enough, we have:

$$\omega_p^{d-1}(C_i, H_i) = p^{1/d} a(H_i) + o(p^{1/d}).$$

Hence we see that:

$$\begin{aligned} \sum_{i=1}^N \sum_{\ell=1}^K \mathbf{M}_{H_i}(\partial E_\ell^i(x)) &\leq p^{1/d} \sum_{i=1}^N \sum_{\ell=1}^K (w_\ell^i)^{\frac{d-1}{d}} a(H_i) \\ &\quad + C(\lambda, d)NK + C(N, K) \left(\varepsilon + o(p^{1/d}) \right). \end{aligned}$$

Inside each cube i we define m_i to be the number of slabs S_ℓ whose intersection with C_i has positive volume:

$$m_i := |\{1 \leq \ell \leq K : |S_\ell \cap C_i| > 0\}| = |\{1 \leq \ell \leq K : w_\ell^i > 0\}|.$$

Since these pieces partition C_i modulo null sets, $\sum_{\ell=1}^K w_\ell^i = 1$. Then using Hölder inequality in each cube and then with all the slabs, we have:

$$\sum_{i=1}^N \sum_{\ell=1}^K (w_\ell^i)^{\frac{d-1}{d}} a(H_i) \leq \sum_{i=1}^N m_i^{\frac{1}{d}} a(H_i) \leq \left(\sum_{i=1}^N m_i \right)^{1/d} \left(\sum_{i=1}^N a(H_i)^{\frac{d}{d-1}} \right)^{\frac{d-1}{d}}.$$

This is indeed an optimal integer allocation problem. Note that the total number of positive-volume slabs S_ℓ^i is at most $K + N - 1$, since the total number of slabs is at most the number of cuts plus one (counting the cuts the cubes already have), which is indeed $K + N - 1$. Then we have:

$$\sum_{i=1}^N \sum_{\ell=1}^K (w_\ell^i)^{\frac{d-1}{d}} a(H_i) \leq (K + N - 1)^{\frac{1}{d}} \left(\sum_{i=1}^N a(H_i)^{\frac{d}{d-1}} \right)^{\frac{d-1}{d}}.$$

Putting everything together, we finish the proof. $\square$

4.4. The upper bound. In this section we prove the general upper bound for manifolds with variable anisotropies. To do this we subdivide the manifold into small cubes and map them into a cube in a long rectangle. Then we use the constructions of the last section and build admissible sweepouts with correct energy bounds.

Theorem 4.7. *Let (M^d, g) be a smooth, compact, connected manifold with smooth boundary (possibly empty), $d \geq 2$, and let F satisfy (H1-H3). Then*

$$\limsup_{p \rightarrow \infty} p^{-1/d} \omega_p^{d-1}(M, F) \leq \left(\int_M a(F(x, \cdot))^{\frac{d}{d-1}} dV_g(x) \right)^{\frac{d-1}{d}}.$$

Proof. We first assume that $\partial M = \emptyset$. Fix $\varepsilon > 0$. Partition M , up to boundaries, into finitely many piecewise smooth regions whose closures lie in small normal-coordinate charts. Cover their coordinate images by grid cubes, retaining the whole cubes meeting each region. For sufficiently fine grids, all retained cubes lie in their coordinate domains, and the total excess volume tends to zero as the grids are refined.

Enumerate the retained coordinate cubes as

$$B_i = b_i + \ell_i[0, 1]^d, \quad i = 1, \dots, N,$$

where $b_i \in \mathbb{R}^d$ and $\ell_i > 0$. For each i , let

$$\varphi_i : U_i \subset \mathbb{R}^d \longrightarrow M$$

be the normal-coordinate parametrization used for B_i , so that $B_i \subset U_i$. The same chart may occur for several indices.

We define

$$C_i = [i-1, i] \times [0, 1]^{d-1}, \quad Q := \bigcup_{i=1}^N C_i = [0, N] \times [0, 1]^{d-1}.$$

Thus C_i is the normalized copy of B_i , placed in the i -th position of the long rectangle Q . The corresponding map back to M is

$$f_i : C_i \longrightarrow M, \quad f_i(y_1, \dots, y_d) := \varphi_i(b_i + \ell_i(y_1 - i + 1, y_2, \dots, y_d)).$$

Indeed, this formula first translates C_i onto $[0, 1]^d$, then rescales it onto B_i , and finally applies the coordinate parametrization φ_i . In particular,

$$f_i(C_i) = \varphi_i(B_i).$$

We set

$$x_i = f_i\left(i - \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right), \quad H_i = F(x_i, \cdot), \quad v_i = \text{Vol}_g(f_i(C_i)),$$

using the orthonormal frame obtained from the coordinate frame at x_i to represent H_i . Here v_i records the volume of the whole coordinate cube in M , whereas each normalized cube C_i has unit Euclidean volume; see Figure 6.

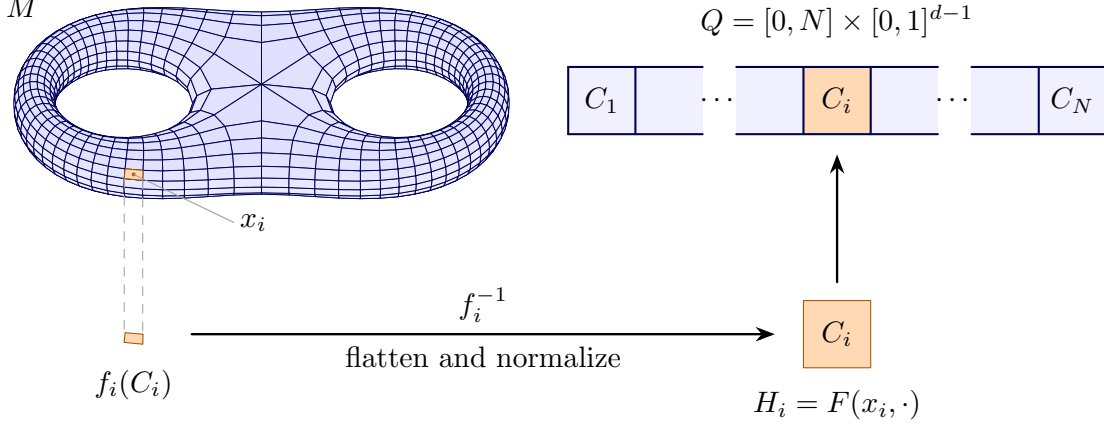


FIGURE 6. The coordinate cubes are flattened, normalized, and arranged as Q . The highlighted cube carries the frozen anisotropy $H_i = F(x_i, \cdot)$.

By choosing the charts sufficiently small and then refining the grids, normal-coordinate estimates and continuity of F give

$$\mathbf{F}((f_i)_\# T) \leq (1 + \varepsilon) v_i^{\frac{d-1}{d}} \mathbf{M}_{H_i}(T)$$

for every rectifiable $(d-1)$ -chain T in C_i . Moreover, continuity of the frozen coefficient and the vanishing excess volume give

$$\sum_{i=1}^N v_i a(H_i)^{\frac{d}{d-1}} \leq \int_M a(F(x, \cdot))^{\frac{d}{d-1}} dV_g(x) + \varepsilon.$$

Fix this cover and $K \geq 1$. Apply the preceding join construction on Q to the frozen integrands $v_i^{(d-1)/d} H_i$. By homogeneity,

$$a\left(v_i^{\frac{d-1}{d}} H_i\right)^{\frac{d}{d-1}} = v_i a(H_i)^{\frac{d}{d-1}}.$$

Transfer the resulting fillings to M by setting

$$D_i := f_i(C_i) \setminus \bigcup_{j < i} f_j(C_j), \quad \tilde{\Psi}_M(z) := \bigcup_{i=1}^N \left(f_i(\tilde{\Psi}(z) \cap C_i) \cap D_i \right).$$

Since the D_i partition M up to null sets, this map is L^1 continuous and satisfies

$$\tilde{\Psi}_M(\tau^{(K)} z) = M \setminus \tilde{\Psi}_M(z).$$

Consequently, $\Psi_M([z]) := \partial[\tilde{\Psi}_M(z)]$ is flat continuous and pulls back the same filling double cover as Ψ . Hence it is a $K(p+1) - 1$ sweepout by Lemma 4.5.

The additional boundaries are supported on the fixed interfaces $\bigcup_i \partial D_i$, whose total area bounds their mass independently of p and K . Their mass measures have no atoms, so these interfaces introduce no concentration of mass. The same property for the remaining pieces follows from that of Ψ and the bi-Lipschitz coordinate maps.

We may therefore apply Proposition 2.1 on M . The freezing estimate and Lemma 4.6 then yield

$$\omega_{K(p+1)-1}^{d-1}(M, F) \leq (1 + \varepsilon) p^{1/d} (K + N - 1)^{1/d} \left(\sum_{i=1}^N v_i a(H_i)^{\frac{d}{d-1}} \right)^{\frac{d-1}{d}} + o(p^{1/d}) + C_{\varepsilon, K},$$

where the cover and K are fixed as $p \rightarrow \infty$.

Divide by $(K(p+1) - 1)^{1/d}$ and let $p \rightarrow \infty$. Monotonicity of the widths extends the resulting estimate to all indices, since

$$\frac{K(p+2) - 1}{K(p+1) - 1} = 1 + \frac{K}{K(p+1) - 1} \rightarrow 1.$$

Thus

$$\limsup_{p \rightarrow \infty} p^{-1/d} \omega_p^{d-1}(M, F) \leq (1 + \varepsilon) \left(1 + \frac{N-1}{K} \right)^{1/d} \left(\sum_{i=1}^N v_i a(H_i)^{\frac{d}{d-1}} \right)^{\frac{d-1}{d}}.$$

Letting $K \rightarrow \infty$ with the cover fixed, and using the weighted Riemann-sum estimate above, gives

$$\limsup_{p \rightarrow \infty} p^{-1/d} \omega_p^{d-1}(M, F) \leq (1 + \varepsilon) \left(\int_M a(F(x, \cdot))^{\frac{d}{d-1}} dV_g(x) + \varepsilon \right)^{\frac{d-1}{d}}.$$

Finally, let $\varepsilon \downarrow 0$.

For a nonempty smooth boundary, embed M as a compact domain in a smooth closed connected manifold $\widehat{M}$, extending g smoothly. Given $\eta > 0$, extend F continuously to a thin exterior collar, preserving positivity and convexity of the normal extension. Such an extension is obtained by transporting the normal integrands along the collar through linear isomorphisms of the tangent spaces. Blend this extension with a small positive constant multiple of the area integrand outside the collar, using a partition of unity depending only on the base point. Convexity and positivity are preserved. The collar can be chosen sufficiently thin and the constant sufficiently small that the resulting integrand $\widehat{F}$ agrees with F on M and satisfies

$$\int_{\widehat{M} \setminus M} a(\widehat{F}(x, \cdot))^{\frac{d}{d-1}} dV_g(x) < \eta.$$

Here boundedness on the collar follows from comparability, and homogeneity of a controls the remaining region.

To compare widths, lift any mass-continuous p -sweepout on $\widehat{M}$ to its filling double cover and restrict each filling E to $E \cap M$. This restriction is L^1 continuous and commutes with complementation in the respective domains. Its relative boundary therefore pulls back the same covering class, and its anisotropic mass is at most that of the original cycle. Indeed, the new trace term lies on ∂M and is ignored in the relative mass. The relative boundary measures inside M are restrictions of the original boundary measures, so no concentration of mass is preserved. Applying Proposition 2.1 on M and then taking the infimum over the original sweepouts yields

$$\omega_p^{d-1}(M, F) \leq \omega_p^{d-1}(\widehat{M}, \widehat{F}).$$

The closed case just proved gives

$$\limsup_{p \rightarrow \infty} p^{-1/d} \omega_p^{d-1}(M, F) \leq \left(\int_M a(F(x, \cdot))^{\frac{d}{d-1}} dV_g(x) + \eta \right)^{\frac{d-1}{d}}.$$

Letting $\eta \downarrow 0$ proves the result with boundary. $\square$

At this point we are ready to prove the full variable Weyl law combining the lower and upper bound:

Proof of Theorem 1.2. The proof follows immediately from combining the lower bound of Corollary 4.2 and the matching upper bound in Theorem 4.7. $\square$

Part 2. Grassmannian equidistribution for nearby anisotropies

5. THE SMOOTH REALIZATION ASSUMPTION AND THE CONCLUSIONS

Throughout this part, (M^d, g) is closed, connected, and smooth, with $3 \leq d \leq 7$. We will keep the metric g fixed and we will instead perturb the anisotropic integrand. We write $G := G_{d-1}(M)$, for the bundle of unoriented tangent hyperplanes. For a varifold V , we write

$$\mathbf{F}(V) := \mathbf{F}(V, M) = \int_G F(x, T) dV(x, T).$$

To prove equidistribution in the Grassmannian bundle, we combine the Weyl law with smooth realization of every width for integrands sufficiently close to area. We assume throughout Part 2 this realization property in Assumption 5.1. Theorem 1.4 proves Assumption 5.1 for $3 \leq d \leq 6$.

Assumption 5.1. There exist $\delta_0 > 0$ and an integer $k_0 \geq 6$, with the following property. For every smooth integrand $F : G \rightarrow (0, \infty)$ satisfying $\|F - 1\|_{C^{k_0}(G)} < \delta_0$ and every $p \geq 1$, there exists an integral varifold

$$V = \sum_{i=1}^s m_i |\Sigma_i|, \quad m_i \in \mathbb{N},$$

such that Σ_i are smooth, closed, embedded, connected, pairwise disjoint F -stationary hypersurfaces and

$$\mathbf{F}(V) = \omega_p^{d-1}(M, F).$$

We can decrease δ_0 so that the normal representations of the integrands in this neighborhood of the area are uniformly strictly elliptic.

The following theorem is the main result of Part 2.

Theorem 5.2. Assume that (M^d, g) satisfies Assumption 5.1. There exist a sequence of smooth positive strictly elliptic integrands $F_j \rightarrow 1$ in $C^\infty(G)$ and a sequence of smooth, closed, embedded, connected F_j -stationary hypersurfaces $\{\Sigma_j\}_j$ such that

$$\lim_{q \rightarrow \infty} \frac{\sum_{j=1}^q \int_{\Sigma_j} h(x, T_x \Sigma_j) dA_g}{\sum_{j=1}^q \text{area}_g(\Sigma_j)} = \int_G h d\Lambda_g \quad \text{for every } h \in C^0(G). \quad (5.1)$$

Each Σ_j can be chosen nondegenerate for F_j and as a component of a smooth F_j -stationary varifold realizing $\omega_{p_j}^{d-1}(M, F_j)$ for some $p_j \geq 1$.

Here nondegeneracy means that the Jacobi operator has trivial kernel on smooth sections of the normal bundle. Combining Theorem 5.2 with Theorem 1.4, we deduce Theorem 1.3.

The proof follows the finite-dimensional deformation and averaging argument of Marques-Neves-Song [40], with the anisotropy as the parameter. We first use rotation invariance and homogeneity of the frozen Weyl constant to identify the limit of averaged derivatives of the Weyl functional. This step does not require differentiability of a at 1 or an explicit formula for a . We then adapt White's transversality argument to variations of F with g fixed. Together with Assumption 5.1, this gives a formula for derivatives of widths in terms

of smooth stationary hypersurfaces. We obtain finite weighted collections approximating any prescribed finite family of moments of Λ_g , and then assemble these collections into the sequence in Theorem 5.2.

6. AVERAGED DERIVATIVES OF THE WEYL FUNCTIONAL

We work in the neighborhood of area from Assumption 5.1. We use the Weyl functional $\mathcal{W}$ and the probability measure Λ_g introduced in (1.5) and (1.2), and set

$$L_g(h) := \mathcal{W}(1) \int_G h d\Lambda_g.$$

Our first goal is to approximate L_g by averaged derivatives of $\mathcal{W}$ at integrands approaching area. We will not assume differentiability of a at 1, and we will rely instead on its rotation invariance and homogeneity.

6.1. Finite-dimensional families of test functions. For $m \geq 0$, let $\mathcal{E}_m \rightarrow M$ be the vector bundle of even spherical harmonics of degrees at most $2m$ on the unit spheres of TM , equipped with the fiberwise $L^2(\sigma_x)$ inner product. We identify its sections with functions on G . The union, over $m \geq 0$, of the spaces of smooth sections of $\mathcal{E}_m$ is uniformly dense in $C(G)$. Indeed, one can first approximate a continuous function by a smooth one, then apply uniform polynomial approximation in local orthonormal frames and combine the approximations with a partition of unity. Taking the even parts gives a finite harmonic expansion in each fiber.

The following lemma provides finite families whose evaluation maps respect the fiberwise inner products. It allows us to average over the fibers without choosing a global orthonormal frame of TM .

Lemma 6.1. *Let $\mathcal{E} \rightarrow M$ be a smooth Euclidean vector bundle of rank r , and let $h_0, \dots, h_s$ be smooth sections. There exist smooth sections $s_1, \dots, s_L$ such that*

$$\sum_{a=1}^L s_a(x) \otimes s_a(x) = \text{Id}_{\mathcal{E}_x} \text{ for every } x \in M, \quad \text{and} \quad h_0, \dots, h_s \in \text{span}_{\mathbb{R}}\{s_1, \dots, s_L\}.$$

In particular, the maps

$$A_x : \mathbb{R}^L \longrightarrow \mathcal{E}_x, \quad A_x t := \sum_{a=1}^L t_a s_a(x),$$

satisfy $A_x A_x^* = \text{Id}_{\mathcal{E}_x}$.

Proof. Choose a finite open cover $U_1, \dots, U_N$ of M such that $\mathcal{E}$ admits a smooth orthonormal frame $e_{i,1}, \dots, e_{i,r}$ on each U_i . Choose smooth functions $\psi_1, \dots, \psi_N$ on M such that

$$\text{spt } \psi_i \subset U_i \quad \text{for } i = 1, \dots, N, \quad \sum_{i=1}^N \psi_i^2 = 1.$$

Since M is compact, we can choose $\eta > 0$ sufficiently small that

$$B := \text{Id}_{\mathcal{E}} - \eta^2 \sum_{j=0}^s h_j \otimes h_j$$

is positive definite. Its positive square root $B^{1/2}$ is a smooth bundle endomorphism. We take $s_1, \dots, s_L$, with $L = Nr + s + 1$, to be the sections

$$\psi_i B^{1/2} e_{i,\alpha} \quad (1 \leq i \leq N, 1 \leq \alpha \leq r), \quad \eta h_0, \dots, \eta h_s,$$

where the sections supported in U_i are extended by zero outside U_i .

By orthonormality of the local frames,

$$\sum_{a=1}^L s_a \otimes s_a = \sum_{i=1}^N \psi_i^2 B + \eta^2 \sum_{j=0}^s h_j \otimes h_j = \text{Id}_{\mathcal{E}}.$$

Moreover, each h_j is a constant multiple of one of the chosen sections. Finally,

$$A_x A_x^* = \sum_{a=1}^L s_a(x) \otimes s_a(x) = \text{Id}_{\mathcal{E}_x},$$

which completes the proof. $\square$

6.2. Averaging the Weyl derivatives. From Lemma 3.5 we know that a is locally Lipschitz. The same argument gives local Lipschitz continuity of $\mathcal{W}$. We now identify the limit of its averaged derivatives near 1 (the area functional).

Proposition 6.2. *Let $h_0 = 1, h_1, \dots, h_s$ be smooth sections of $\mathcal{E}_m$. There exist a finite-dimensional subspace $U \subset C^\infty(G)$ containing these sections and a surjective linear map $A : \mathbb{R}^L \rightarrow U$ such that*

$$w(v) := \mathcal{W}(1 + Av)$$

is well-defined and Lipschitz in a neighborhood of 0. Moreover, for every nonnegative smooth radial function ρ compactly supported in the unit ball of $\mathbb{R}^L$ with $\int_{\mathbb{R}^L} \rho(t) dt = 1$, we have

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^L} \rho(t) Dw(\varepsilon t)[b] dt = L_g(Ab) \quad \text{for every } b \in \mathbb{R}^L. \quad (6.1)$$

Consequently, $L_g|_U$ belongs to the closed convex hull of the limits of derivatives of $\mathcal{W}$ along $1 + U$ at differentiability points approaching 1. The corresponding integrands converge to 1 in $C^\infty(G)$.

Proof. Let $s_1, \dots, s_L$ be the sections given by Lemma 6.1, and set

$$U := \text{span}_{\mathbb{R}}\{s_1, \dots, s_L\}, \quad At := \sum_{a=1}^L t_a s_a.$$

Then U contains $h_0, \dots, h_s$, and A is surjective. Since A has smooth image and finite-dimensional domain, $1 + Av$ belongs to the prescribed neighborhood of area for v sufficiently small. The local Lipschitz bound for $\mathcal{W}$ gives the corresponding bound for w .

The evaluation maps $A_x t = (At)(x, \cdot)$ satisfy $A_x A_x^* = \text{Id}_{\mathcal{E}_{m,x}}$. Since ρ is radial, the pushforward of $\rho(t) dt$ under A_x has a smooth density which is the same in every fiber after orthogonal identification. Indeed, in orthonormal coordinates A_x is an orthogonal projection, and this density is obtained by integrating ρ over the remaining coordinates. In particular, the projected measure is invariant under the action induced by $O(d)$.

We evaluate a in orthonormal frames, as in the definition of $\mathcal{W}$, and take its derivatives within the finite-dimensional fibers $\mathcal{E}_{m,x}$. By the local Lipschitz bound, a is differentiable almost everywhere in each fiber near 1. Since A_x is surjective, Fubini's theorem implies that, for almost every t in the support of ρ , a is differentiable at $1 + \varepsilon A_x t$ for almost every x . The uniform Lipschitz bound allows differentiation under the integral. Applying the chain rule, we obtain

$$Dw(\varepsilon t)[b] = \int_M \left(\frac{a(1 + \varepsilon A_x t)}{w(\varepsilon t)} \right)^{1/(d-1)} Da(1 + \varepsilon A_x t)[A_x b] dV_g(x) \quad (6.2)$$

for almost every t .

By rotation invariance of a and of the projected measure, the linear functional

$$h \mapsto \int_{\mathbb{R}^L} \rho(t) Da(1 + \varepsilon A_x t)[h] dt \quad \text{on } \mathcal{E}_{m,x}$$

is invariant under $O(d)$. Averaging h over $O(d)$ gives its mean times the constant function. Hence

$$\int_{\mathbb{R}^L} \rho(t) Da(1 + \varepsilon A_x t)[h] dt = \left(\int_{G_{d-1}(T_x M)} h d\sigma_x \right) \int_{\mathbb{R}^L} \rho(t) Da(1 + \varepsilon A_x t)[1] dt.$$

On the other hand, homogeneity gives

$$Da(H)[1] = a(H) - Da(H)[H - 1]$$

at every differentiability point. Since Da is locally bounded and $a(1) = a_d$, it follows that

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^L} \rho(t) Da(1 + \varepsilon A_x t)[h] dt = a_d \int_{G_{d-1}(T_x M)} h d\sigma_x. \quad (6.3)$$

The convergence is uniform in x and for h in the unit balls of the fibers.

Moreover, $w(0) = a_d \text{Vol}_g(M)^{(d-1)/d}$, so

$$\left(\frac{a(1 + \varepsilon A_x t)}{w(\varepsilon t)} \right)^{1/(d-1)} \longrightarrow \text{Vol}_g(M)^{-1/d}$$

uniformly for $x \in M$ and $t \in \text{supp } \rho$. Using the uniform bound for Da , we may therefore replace this factor in the average of (6.2) by $\text{Vol}_g(M)^{-1/d}$ with an error tending to zero. Equation (6.3) then gives

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^L} \rho(t) Dw(\varepsilon t)[b] dt \\ = a_d \text{Vol}_g(M)^{-1/d} \int_M \int_{G_{d-1}(T_x M)} (Ab)(x, T) d\sigma_x(T) dV_g(x) = L_g(Ab). \end{aligned}$$

This proves (6.1).

To prove the convex hull assertion, recall that Dw is bounded near 0. If $L_g \circ A$ were outside the closed convex hull of the limits of $Dw(v)$ as $v \rightarrow 0$ through differentiability points, separation and boundedness would give $b \in \mathbb{R}^L$ and $\delta > 0$ such that

$$Dw(v)[b] \leq L_g(Ab) - \delta$$

at every differentiability point sufficiently close to 0. Integrating this inequality with $v = \varepsilon t$ against $\rho(t) dt$ contradicts (6.1).

Since A is surjective, differentiability of w at v is equivalent to differentiability of $\mathcal{W}|_{1+U}$ at $1 + Av$, and the derivatives are related by composition with A . Thus the preceding conclusion gives the asserted convex hull property for the derivatives of $\mathcal{W}$ restricted to $1 + U$. Finally, for every k ,

$$\|Av\|_{C^k} \leq \|A\|_{\mathbb{R}^L \rightarrow C^k} |v| \longrightarrow 0 \quad \text{as } v \rightarrow 0,$$

which proves the last assertion. $\square$

6.3. Passing to the volume spectrum. For the map A given by Proposition 6.2, set

$$u_p(v) := \tilde{\omega}_p^{d-1}(M, 1 + Av).$$

The Weyl law gives $u_p \rightarrow w$ pointwise. Comparison of the integrands and the uniform bound for $\tilde{\omega}_p^{d-1}(M, 1)$ give a Lipschitz bound for u_p , independent of p , on a sufficiently small parameter ball. Hence the convergence is uniform on compact subsets. The following lemma allows us to pass to the limit in averages of the derivatives.

Lemma 6.3. *Let A and ρ be as in Proposition 6.2. For every sufficiently small fixed $\varepsilon > 0$, we have*

$$\lim_{p \rightarrow \infty} \int_{\mathbb{R}^L} \rho(t) Du_p(\varepsilon t)[b] dt = \int_{\mathbb{R}^L} \rho(t) Dw(\varepsilon t)[b] dt \quad \text{for every } b \in \mathbb{R}^L. \quad (6.4)$$

For each u_p and for w , the corresponding averaged derivative can be approximated, uniformly for $|b| \leq 1$, by a convex combination of at most $L + 1$ derivatives evaluated at points of $\varepsilon \operatorname{supp} \rho$. These points may be chosen in any prescribed subset of full Lebesgue measure in the parameter ball on which the corresponding function is differentiable.

Proof. Since u_p is Lipschitz and ρ is smooth and compactly supported, integration by parts gives

$$\int_{\mathbb{R}^L} \rho(t) Du_p(\varepsilon t)[b] dt = -\frac{1}{\varepsilon} \int_{\mathbb{R}^L} u_p(\varepsilon t) D\rho(t)[b] dt.$$

The same identity holds for w . Subtracting the two identities, we bound the difference of the averaged derivatives by

$$\frac{|b|}{\varepsilon} \sup_{t \in \operatorname{supp} \rho} |u_p(\varepsilon t) - w(\varepsilon t)| \int_{\mathbb{R}^L} |D\rho(t)| dt.$$

Since ε is fixed, this tends to zero uniformly for $|b| \leq 1$, proving (6.4).

Finally, each averaged derivative belongs to the closed convex hull of the derivatives evaluated at points of $\varepsilon \operatorname{supp} \rho$ in the prescribed full-measure set. It can therefore be approximated uniformly for $|b| \leq 1$ by a finite convex combination of such derivatives. Since the space of linear functionals on $\mathbb{R}^L$ has dimension L , Carathéodory's theorem reduces each combination to at most $L + 1$ terms. $\square$

7. GEOMETRIC REALIZATION AND EQUIDISTRIBUTION

We now use the smooth realizations in Assumption 5.1 and deformations of the integrand to represent the averaged spectral derivatives by stationary hypersurfaces.

7.1. Deformations of stationary hypersurfaces. We adapt White's deformation argument [51] by varying the integrand while keeping the metric fixed. All perturbations remain in the neighborhood specified in Assumption 5.1. We first record the variation formulas that we will use.

Lemma 7.1. *Let F be a smooth strictly elliptic integrand, and let $\Sigma \subset M$ be a smooth, closed, embedded hypersurface. For a smooth normal variation, write its velocity locally as $u\nu$, where ν is a unit normal. Then*

$$\delta \mathbf{F}(|\Sigma|)[u\nu] = \int_{\Sigma} u H_F dA_g, \quad H_F = \operatorname{div}_{\Sigma} D_{\xi} \varphi_x(\nu) + \langle \nabla_x \varphi_x(\nu), \nu \rangle. \quad (7.1)$$

Here ∇_x denotes differentiation in the base point while parallel transporting ξ with respect to the Levi-Civita connection, and $\operatorname{div}_{\Sigma}$ is the tangential trace of the ambient covariant derivative.

If Σ is F -stationary, the linearization J_F of H_F under normal variations has the local expression

$$J_F u = -\operatorname{div}_{\Sigma}(A_F \nabla_{\Sigma} u) + c_F u, \quad A_F(x) := D_{\xi}^2 \varphi_x(\nu)|_{T_x \Sigma}, \quad (7.2)$$

where c_F is a smooth function on Σ . The operator is elliptic and formally self-adjoint with respect to dA_g . For every $0 < \alpha < 1$, it is Fredholm of index zero from $C^{2,\alpha}$ sections of the normal bundle to $C^{0,\alpha}$ sections. For $F = 1$, it reduces to

$$J_F u = -\Delta_{\Sigma} u - (|A_{\Sigma}|^2 + \operatorname{Ric}_g(\nu, \nu))u.$$

Finally, let $f \in C^{\infty}(M)$ and set $F_s(x, T) = e^{sf(x)} F(x, T)$. On an F -stationary hypersurface,

$$\left. \frac{d}{ds} \right|_{s=0} H_{F_s} = \langle D_{\xi} \varphi_x(\nu), \nabla f \rangle. \quad (7.3)$$

If $f|_{\Sigma} = 0$, the right-hand side is $F(x, T_x \Sigma) \partial_{\nu} f$. All these formulas also apply to one-sided hypersurfaces, with the scalar expressions interpreted using local unit normals.

Proof. Set $H = \operatorname{div}_\Sigma \nu$. Under a normal variation with velocity $u\nu$, the derivatives of the normal and the area element are $-\nabla_\Sigma u$ and $uH dA_g$, respectively. Thus

$$\delta \mathbf{F}(|\Sigma|)[u\nu] = \int_\Sigma (u \langle \nabla_x \varphi_x(\nu), \nu \rangle - \langle D_\xi \varphi_x(\nu), \nabla_\Sigma u \rangle + uH \varphi_x(\nu)) dA_g.$$

By homogeneity, $\langle D_\xi \varphi_x(\nu), \nu \rangle = \varphi_x(\nu)$. Integrating the tangential part of $D_\xi \varphi_x(\nu)$ by parts therefore proves (7.1).

The second-order terms in the normal linearization of H_F agree with those of the operator $-\operatorname{div}_\Sigma(A_F \nabla_\Sigma u)$. At an F -stationary hypersurface, symmetry of the second variation makes J_F formally self-adjoint with respect to dA_g . The divergence-form operator is also formally self-adjoint, so their difference has no first-order term. This proves (7.2). Strict ellipticity of F gives ellipticity of J_F . Since Σ is closed, elliptic theory gives the Fredholm property, and self-adjointness gives index zero. When $F = 1$, we have $H_F = H$, and the stated expression for J_F is the usual normal variation formula for mean curvature.

The normal representation of F_s is $e^{sf(x)} \varphi_x(\xi)$. Substituting this into (7.1) gives

$$H_{F_s} = e^{sf} (H_F + s \langle D_\xi \varphi_x(\nu), \nabla f \rangle).$$

Since $H_F = 0$, differentiation at $s = 0$ proves (7.3). If $f|_\Sigma = 0$, then $\nabla f = (\partial_\nu f)\nu$ along Σ , and homogeneity gives

$$\langle D_\xi \varphi_x(\nu), \nabla f \rangle = \varphi_x(\nu) \partial_\nu f = F(x, T_x \Sigma) \partial_\nu f.$$

Finally, φ_x is even. Replacing ν by $-\nu$ changes the signs of u , H_F , and $J_F u$. Hence uH_F and the normal fields $H_F \nu$ and $(J_F u)\nu$ are independent of the local choice of unit normal. This proves that the formulas also apply when Σ is one-sided. $\square$

Proposition 7.2. *Fix an integer $q \geq k_0$ and $0 < \alpha < 1$. Consider pairs (F, Σ) , where F is a C^q integrand in the strictly elliptic neighborhood of area fixed above and Σ is a closed embedded $C^{2,\alpha}$ F -stationary hypersurface. The space of these pairs is a separable C^{q-3} Banach manifold admitting a countable atlas, and the projection*

$$(F, \Sigma) \longmapsto F$$

is Fredholm of index zero. Its critical points are precisely the pairs for which J_F has a nontrivial kernel on $C^{2,\alpha}$ sections of the normal bundle.

Within the same neighborhood, the smooth integrands for which every closed embedded stationary hypersurface is nondegenerate form a residual set in the C^∞ topology. Moreover, every smooth family $t \mapsto F(t)$, parametrized by an open subset of $\mathbb{R}^b$, can be perturbed arbitrarily slightly in the C^∞ topology to a family transverse to the projection, taking $q \geq \max\{k_0, b + 5\}$. For such a family, every closed embedded $F(t)$ -stationary hypersurface is nondegenerate for almost every t .

All assertions hold for both two-sided and one-sided hypersurfaces, with J_F acting on sections of the normal bundle.

Proof. We use the local construction in the proof of [51, Theorem 2.1], with the integrand as parameter. The variation formulas and Fredholm properties in Lemma 7.1 remain valid for C^q integrands and $C^{2,\alpha}$ stationary hypersurfaces, by the same local computations and elliptic regularity.

Choose a fixed smooth reference hypersurface and represent nearby hypersurfaces as normal graphs of $C^{2,\alpha}$ sections. After identifying the normal bundles and pulling back to the reference hypersurface, the stationarity equation defines a C^{q-3} map into the fixed space of $C^{0,\alpha}$ normal sections. At a stationary graph, its derivative in the graph variable is identified with J_F : the graph velocity is first projected onto the normal bundle of the current hypersurface, and the resulting mean-curvature variation is pulled back to the reference hypersurface. The terms obtained by differentiating these identifications vanish

because $H_F = 0$. We first show that variations of the integrand complement the range of this operator.

By self-adjointness, it suffices to show that no nonzero $w \in \ker J_F$ is orthogonal to all variations of H_F induced by changing F . Choose a patch where a unit normal is defined and w is nonzero, and represent w by its scalar coefficient with respect to this normal. Let χ be a nonnegative smooth cutoff supported in the patch, with $\chi w \not\equiv 0$. In a tubular neighborhood of the patch, choose an ambient function f satisfying $f|_\Sigma = 0$, and $\partial_\nu f = \chi w$, and extend it by zero away from that neighborhood. By (7.3),

$$\int_\Sigma w \frac{d}{ds} \Big|_{s=0} H_{e^{sf} F} dA_g = \int_\Sigma \chi F(x, T_x \Sigma) w^2 dA_g > 0.$$

The scalar expressions in this identity are supported in the chosen patch. If Σ has only finite regularity, we approximate f in C^1 by smooth ambient functions supported in the same neighborhood. The strict inequality persists, since the variation of H_F depends continuously on f in C^1 .

Thus the integrand variations span the cokernel of J_F . Since this cokernel is finite-dimensional, finitely many smooth variations suffice. The full linearization is therefore surjective with complemented kernel, and the implicit function theorem gives the local solution manifold. The kernel of the derivative of the projection is $\ker J_F$, while its range has codimension $\dim \ker J_F$. This proves the index and critical point assertions. Since the preceding perturbations are supported in a single patch, the argument also applies when the normal bundle is nontrivial.

We next justify the existence of a countable atlas for the space of pairs (F, Σ) . For each diffeomorphism type, choose a countable family of smooth reference embeddings into M that is dense in the C^3 topology. Elliptic regularity gives $C^{3,\alpha}$ local graph representations for every stationary hypersurface in the space. Each such hypersurface can therefore be represented as a normal graph over a sufficiently close reference hypersurface from one of these countable families.

In each reference chart, the stationary graph sections belong to the closure of the smooth sections in $C^{2,\alpha}$. This closure is a separable metric space, as is $C^q(G)$. The solution set in the chart is therefore second countable in the induced topology. Since there are only countably many diffeomorphism types and reference embeddings, these charts give a countable open cover of the solution space. Consequently, the solution manifold is second countable and admits a countable Banach manifold atlas.

We next prove the assertions about smooth families. Fix the parameter dimension b and take $q \geq \max\{k_0, b + 5\}$. At a fixed solution, split the normal sections into $\ker J_F$ and its orthogonal complement. Solving the equation in the complement by the implicit function theorem reduces stationarity locally to

$$K(F, z) = 0, \quad z \in \mathbb{R}^r, \quad K(F, z) \in \mathbb{R}^r,$$

where r is the dimension of the Jacobi kernel at the fixed solution. The map K is C^{q-3} , and its full derivative is surjective at every zero in a sufficiently small neighborhood. A family $t \mapsto F(t)$ is transverse to the projection precisely when $D_{(t,z)}(K(F(t), z)) : \mathbb{R}^b \times \mathbb{R}^r \rightarrow \mathbb{R}^r$ is surjective at every zero.

We verify this condition on compact subsets of the t and z domains, restricting $F(t)$ to a closed C^q ball contained strictly inside the integrand domain of the chart. For any fixed family, the corresponding set of zeros in (t, z) is compact. Surjectivity at these zeros is therefore open under C^1 perturbations of the family with values in C^q .

To prove density, first choose slightly larger restrictions of the same kind. The preceding cokernel argument and compactness give finitely many smooth functions $f_1, \dots, f_N$ such

that, for

$$F_{\mathbf{a}}(t, x, T) = \exp \left(\sum_{i=1}^N a_i f_i(x) \right) F(t, x, T), \quad \mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^N,$$

the derivative of $K(F_{\mathbf{a}}(t), z)$ with respect to $(\mathbf{a}, t, z)$ is surjective near the relevant zeros. The larger restrictions ensure that every zero in the smaller region remains covered after a sufficiently small perturbation. A cutoff in t , equal to one on the larger compact parameter set, allows these perturbations to be supported in a compact subset of the parameter domain.

The resulting zero set has dimension $b + N$. Since $q - 3 > b$, Sard's theorem applied to its projection onto $\mathbb{R}^N$ shows that arbitrarily small choices of $\mathbf{a}$ give a transverse family on the restricted region. This is the local parametric transversality argument of [48]. Because all f_i are smooth, the perturbation can be made arbitrarily small in any prescribed finite collection of C^∞ seminorms. It also preserves strict ellipticity and the fixed C^{k_0} neighborhood.

We equip the space of smooth families taking values in the prescribed neighborhood with the C^∞ topology on compact subsets of the parameter domain. This is a Baire space. Indeed, a countable compact exhaustion of the parameter domain expresses the condition of taking values in the prescribed neighborhood as a countable intersection of open conditions in the Fréchet space of smooth functions on the product of the parameter domain with G . The resulting space is therefore completely metrizable.

Using the countable atlas, we choose countably many nested restrictions of the preceding kind whose interiors cover all solutions. Each localized transversality condition is open and dense in this space of smooth families, by the preceding perturbation argument. Baire's theorem therefore gives a residual, and hence dense, set of families transverse in every chart. Taking $b = 0$ proves the residual assertion for smooth integrands.

Finally, for a transverse b -parameter family, the space of pairs (t, Σ) with Σ $F(t)$ -stationary is a b -dimensional C^{q-3} manifold with a countable atlas. Sard's theorem implies that almost every t is a regular value of the projection $(t, \Sigma) \mapsto t$. The kernel of its derivative at (t, Σ) is precisely $\ker J_{F(t)}$. Since the domain and target have the same dimension, regularity forces this kernel to vanish. Thus every closed embedded stationary hypersurface is nondegenerate at almost every parameter. $\square$

7.2. Derivatives of the widths. We now combine Assumption 5.1 with Proposition 7.2 to express the derivatives of the widths in terms of their realizing hypersurfaces. We first record an elementary lemma.

Lemma 7.3. *Let $\Omega \subset \mathbb{R}^N$ be open, and let $w : \Omega \rightarrow \mathbb{R}$ be locally Lipschitz. Suppose that there are countably many open sets $U_\ell \subset \Omega$ and functions $e_\ell \in C^1(U_\ell)$ such that, for almost every $t \in \Omega$,*

$$w(t) = e_\ell(t) \quad \text{for some } \ell \text{ with } t \in U_\ell.$$

Then, for almost every $t \in \Omega$, such an ℓ can be chosen so that $Dw(t) = De_\ell(t)$.

Proof. For each ℓ , set $E_\ell := \{t \in U_\ell : w(t) = e_\ell(t)\}$. Almost every point of E_ℓ is a density point of E_ℓ at which w is differentiable. At any such point t , we have $D(w - e_\ell)(t) = 0$. Indeed, a nonzero differential would imply that $w - e_\ell$ is nonzero in a cone with vertex t , contradicting the density-one property of E_ℓ . Since the collection is countable and $\bigcup_\ell E_\ell$ has full measure in Ω , the conclusion follows. $\square$

Proposition 7.4. *Suppose that Assumption 5.1 holds, and let $t \mapsto F(t)$ be a smooth family of smooth integrands in the realization neighborhood of area, defined near a compact cube in $\mathbb{R}^N$. After an arbitrarily small perturbation in the C^∞ topology, there is a full-measure set of parameters in the cube with the following property.*

For every $p \geq 1$, the function $t \mapsto \omega_p^{d-1}(M, F(t))$ is differentiable at t , and there is a realizing integral varifold $V_{p,t}$ with smooth nondegenerate connected components such that

$$D_t \omega_p^{d-1}(M, F(t))[v] = \int_G D_t F(t)[v](x, T) dV_{p,t}(x, T) \quad \text{for every } v \in \mathbb{R}^N. \quad (7.4)$$

The full-measure set is independent of p .

Proof. Choose $q \geq \max\{k_0, N + 5\}$ and apply Proposition 7.2 to perturb the family to one transverse to the projection. We keep the perturbation inside the neighborhood in Assumption 5.1. For the resulting family, all closed embedded $F(t)$ -stationary hypersurfaces are nondegenerate for almost every parameter t .

By Proposition 7.2, the derivative of the projection onto the integrand space is an isomorphism at every nondegenerate stationary hypersurface. The inverse function theorem therefore gives a neighborhood of each such pair on which the projection is a diffeomorphism onto an open set of integrands. Since the solution manifold admits a countable atlas, these neighborhoods have a countable subcover. Composing the corresponding local inverses with $t \mapsto F(t)$ gives countably many C^1 branches, defined on open parameter sets, which cover all nondegenerate connected stationary hypersurfaces in the family.

A varifold realizing a width may have several connected components, each with a positive integer multiplicity. We therefore consider all finite collections of these branches, assigning a positive integer multiplicity to each member. Each collection is defined on the intersection of the parameter domains of its branches, restricted to the open set where the corresponding hypersurfaces are pairwise disjoint. Adding their energies with the chosen multiplicities gives a C^1 function. There are countably many such choices, so the resulting energy functions form a countable collection, which we denote by e_ℓ . At every parameter where all stationary hypersurfaces are nondegenerate, each realization supplied by Assumption 5.1 is represented by one of these collections.

Fix $p \geq 1$. Comparison of the integrands shows that $t \mapsto \omega_p^{d-1}(M, F(t))$ is locally Lipschitz. By the preceding paragraph, it agrees almost everywhere with at least one of the functions e_ℓ . Lemma 7.3 therefore gives, at almost every parameter, a realizing collection whose energy has the same differential as the width. Let $V_{p,t}$ be the varifold determined by this collection at t .

When differentiating its energy, the contribution from the motion of each hypersurface vanishes by stationarity. Only the derivative of the integrand remains, giving (7.4). Since the lemma identifies the full differentials, the same varifold $V_{p,t}$ gives this identity for every $v \in \mathbb{R}^N$.

Finally, intersect these full-measure sets over $p \geq 1$ and with the full-measure set where all stationary hypersurfaces are nondegenerate. This gives all the conclusions on a single full-measure set of parameters. $\square$

7.3. Finite stationary collections. We apply Proposition 7.4 to a small perturbation of the affine family. The following lemma controls the resulting error in the averaged derivatives.

Lemma 7.5. *Fix A and ρ as above, $k \geq k_0$, and a compact parameter cube containing $\text{supp } \rho$. Let $\varepsilon > 0$ be sufficiently small that*

$$H(t) := 1 + \varepsilon A t$$

belongs to the neighborhood in Assumption 5.1 on a neighborhood of this cube. For every sufficiently small $\eta > 0$, there exists a smooth family $\tilde{H}$ satisfying the conclusions of Proposition 7.4, with

$$\|\tilde{H} - H\|_{C_t^1 C_G^k} < \eta.$$

The realizing varifolds $V_{p,t}$ given by that proposition can be chosen so that $t \mapsto \int_G h dV_{p,t}$ is measurable for every $h \in C(G)$ and every $p \geq 1$. Moreover,

$$\left| \int \rho(t) \left(p^{-1/d} \int_G Ab dV_{p,t} - Du_p(\varepsilon t)[b] \right) dt \right| \leq C \frac{\eta}{\varepsilon} |b| \quad \text{for every } b \in \mathbb{R}^L, \quad (7.5)$$

where C is independent of p and η .

Proof. Apply Proposition 7.4 to choose $\tilde{H}$ with the required smallness. We first explain how to choose the realizing varifolds measurably.

For each p , enumerate the finite collections of stationary branches constructed in the proof of that proposition. At each parameter in the full-measure set given there, choose the first collection whose energy and differential agree with those of the width. The sets on which these identities hold are measurable: the branch energies are C^1 , the width is locally Lipschitz, and equality of differentials can be checked in the finitely many coordinate directions. For each $h \in C(G)$, its integral over any fixed branch collection varies continuously with the parameter. Thus the selected varifolds have the asserted measurability. On the exceptional null set, set $V_{p,t} = 0$.

Write

$$v_p(t) := \tilde{\omega}_p^{d-1}(M, \tilde{H}(t)).$$

Since the integrands lie between fixed positive constants, comparison of the widths and the isotropic upper bound give

$$p^{-1/d} \|V_{p,t}\|(M) \leq C, \quad \sup_t |v_p(t) - u_p(\varepsilon t)| \leq C\eta.$$

The constants are independent of p and η .

By (7.4),

$$Dv_p(t)[b] = p^{-1/d} \int_G D_t \tilde{H}(t)[b] dV_{p,t}$$

for almost every t and every $b \in \mathbb{R}^L$. Since $D_t H(t)[b] = \varepsilon Ab$, the assumed smallness and the mass bound imply

$$\left| \frac{1}{\varepsilon} Dv_p(t)[b] - p^{-1/d} \int_G Ab dV_{p,t} \right| \leq C \frac{\eta}{\varepsilon} |b|.$$

On the other hand, integration by parts gives

$$\frac{1}{\varepsilon} \int \rho(t) Dv_p(t)[b] dt - \int \rho(t) Du_p(\varepsilon t)[b] dt = -\frac{1}{\varepsilon} \int (v_p(t) - u_p(\varepsilon t)) D\rho(t)[b] dt.$$

Consequently,

$$\left| \frac{1}{\varepsilon} \int \rho(t) Dv_p(t)[b] dt - \int \rho(t) Du_p(\varepsilon t)[b] dt \right| \leq C \frac{\eta}{\varepsilon} |b| \int |\rho(t)| dt.$$

Combining the last two estimates and using $\rho \geq 0$ and $\int \rho = 1$ proves (7.5). $\square$

Proposition 7.6. *Suppose that Assumption 5.1 holds. There exist integers $p_m \uparrow \infty$ and, for every $m \geq 1$, finite collections of smooth anisotropies $F_{m,j}$ in the realization neighborhood, integral varifolds $V_{m,j}$, and weights $\lambda_{m,j} \geq 0$, indexed by $1 \leq j \leq N_m$, with*

$$\sum_{j=1}^{N_m} \lambda_{m,j} = 1,$$

such that each $V_{m,j}$ realizes $\omega_{p_m}^{d-1}(M, F_{m,j})$ and its connected components are smooth, closed, embedded, $F_{m,j}$ -stationary and nondegenerate. Moreover,

$$\max_{1 \leq j \leq N_m} \|F_{m,j} - 1\|_{C^k(G)} \longrightarrow 0 \quad \text{for every } k,$$

and the measures

$$\mu_m := p_m^{-1/d} \sum_{j=1}^{N_m} \lambda_{m,j} V_{m,j}$$

satisfy $\mu_m \rightharpoonup \mathcal{W}(1)\Lambda_g$ on G . In particular,

$$\mu_m(G) \longrightarrow \mathcal{W}(1) > 0, \quad \frac{\mu_m}{\mu_m(G)} \rightharpoonup \Lambda_g.$$

Proof. Choose a uniformly dense sequence $h_0 = 1, h_1, h_2, \dots$ in $C(G)$ consisting of smooth functions with finite fiberwise harmonic expansions.

Fix $m \geq 1$. Apply Proposition 6.2 to $h_0, \dots, h_m$, taking a harmonic bundle containing all these functions. Write $A : \mathbb{R}^L \rightarrow C^\infty(G)$ for the resulting map, and choose $b_0, \dots, b_m \in \mathbb{R}^L$ such that $Ab_\ell = h_\ell$ for $0 \leq \ell \leq m$. Choose a smooth radial probability density ρ supported in the unit ball of $\mathbb{R}^L$, and fix a compact parameter cube containing its support in its interior. All these choices, as well as the corresponding functions w and u_p , depend on m ; we suppress this dependence in the notation. All suprema below are taken over the chosen cube.

First choose $\varepsilon > 0$ sufficiently small that

$$\left| \int_{\mathbb{R}^L} \rho(t) Dw(\varepsilon t)[b_\ell] dt - L_g(h_\ell) \right| < \frac{1}{m} \quad \text{for } 0 \leq \ell \leq m,$$

and

$$\sup_t \|\varepsilon At\|_{C^{\max\{k_0, m\}}(G)} < \min \left\{ \frac{\delta_0}{2}, \frac{1}{2m} \right\}.$$

By Lemma 7.5, we can choose a smooth perturbation $\tilde{H}$ of $t \mapsto 1 + \varepsilon At$ such that

$$\sup_t \|\tilde{H}(t) - 1\|_{C^{\max\{k_0, m\}}(G)} < \min \left\{ \delta_0, \frac{1}{m} \right\},$$

and the error in (7.5) is less than $1/m$ in each of the directions $b_0, \dots, b_m$. Since that estimate is uniform in p , the same perturbation gives this bound for every $p \geq 1$.

Keeping ε and $\tilde{H}$ fixed, choose p_m sufficiently large, with $p_m > p_{m-1}$ when $m > 1$, that the error in (6.4) is less than $1/m$ in the same directions. Combining the three estimates gives

$$\left| \int_{\mathbb{R}^L} \rho(t) p_m^{-1/d} \int_G h_\ell dV_{p_m, t} dt - L_g(h_\ell) \right| < \frac{3}{m} \quad \text{for } 0 \leq \ell \leq m.$$

By Lemma 7.5, the map

$$t \longmapsto \left(p_m^{-1/d} \int_G h_\ell dV_{p_m, t} \right)_{\ell=0}^m$$

is bounded and measurable. Its average against $\rho(t) dt$ belongs to the closed convex hull of its values at parameters in $\text{supp } \rho$ where the varifolds realize the widths and have nondegenerate components. By Carathéodory's theorem, this average can be approximated within $1/m$ in each coordinate by a convex combination of at most $m+2$ such values.

Let $t_1, \dots, t_{N_m}$ be the selected parameters and $\lambda_{m,1}, \dots, \lambda_{m,N_m}$ the corresponding weights. Set

$$F_{m,j} := \tilde{H}(t_j), \quad V_{m,j} := V_{p_m, t_j} \quad \text{for } 1 \leq j \leq N_m.$$

These varifolds have the required realization and nondegeneracy properties, and the corresponding measure μ_m satisfies

$$\left| \int_G h_\ell d\mu_m - L_g(h_\ell) \right| \leq \frac{4}{m} \quad \text{for } 0 \leq \ell \leq m.$$

Recall that

$$L_g(h) = \mathcal{W}(1) \int_G h d\Lambda_g.$$

Taking $h_0 = 1$ therefore gives $\mu_m(G) \rightarrow \mathcal{W}(1)$. The measures μ_m have uniformly bounded mass, so density of the functions h_ℓ extends the convergence to every $h \in C(G)$. This proves the asserted weak convergence. Since $\mathcal{W}(1) > 0$, the normalized measures converge to Λ_g as well.

Finally, for every fixed k and all sufficiently large m , the estimate for $\tilde{H}$ gives

$$\max_{1 \leq j \leq N_m} \|F_{m,j} - 1\|_{C^k(G)} < \frac{1}{m}.$$

This proves the convergence of the anisotropies. $\square$

7.4. Proof of equidistribution. We now pass from the finite weighted collections of Proposition 7.6 to a single sequence of hypersurfaces.

Lemma 7.7. *There exists a sequence of connected components of the varifolds $V_{m,j}$, with repetitions allowed, whose cumulative area measures, normalized by their total mass, converge weakly to Λ_g on G . Moreover, the total area diverges, and the indices m of the selected components tend to infinity.*

Proof. Recall that $\mu_m/\mu_m(G) \rightharpoonup \Lambda_g$. For every m , discard the components with zero coefficients and approximate the remaining coefficients in $\sum_j \lambda_{m,j} V_{m,j}$ by positive rational numbers, so that the corresponding normalized measure changes by at most $1/m$ in total variation. Clearing denominators gives a finite list B_m of hypersurfaces, with repetitions, whose normalized area measures converge weakly to Λ_g .

Fix these lists and concatenate sufficiently many copies of each B_m , choosing the number of repetitions so that the total area accumulated at the end of stage m is at least m times both the area of all preceding stages and the area of B_{m+1} , and is at least m .

At the end of stage m , the contribution of all preceding stages is at most $1/m$ of the total area. Hence the normalized measures converge to Λ_g at these endpoints.

We claim that convergence holds for the entire sequence, and not only at the end of each stage. Indeed, during stage $m+1$, the normalized area measure of the preceding stages together with the completed copies of B_{m+1} is a convex combination of two measures converging to Λ_g . The possible unfinished copy has area at most $1/m$ of the area already accumulated, and therefore changes the normalized integral of any $h \in C(G)$ by at most $2\|h\|_\infty/m$.

Since each stage is finite, the indices m tend to infinity. The construction also ensures that the total area diverges. $\square$

Proof of Theorem 5.2. By Proposition 7.6 and Lemma 7.7, we obtain a sequence of smooth, closed, embedded, connected hypersurfaces $\{\Sigma_j\}_j$ satisfying (5.1), with divergent total area. Each Σ_j is a nondegenerate component of some $V_{m,i}$ realizing the width $\omega_{p_m}^{d-1}(M, F_{m,i})$. We denote by F_j the corresponding anisotropy. In particular, Σ_j is F_j -stationary. Since the indices m tend to infinity, Proposition 7.6 also gives $F_j \rightarrow 1$ in $C^\infty(G)$.

Finally, suppose that only finitely many distinct hypersurfaces occur in the sequence. The union of their lifts to G is a closed set of Λ_g -measure zero, since its projection to M has zero d -dimensional volume. Since all the measures appearing on the left-hand side of (5.1) are supported on this set, they cannot converge weakly to Λ_g . This contradiction completes the proof. $\square$

Part 3. Smooth realization of the anisotropic widths

8. INTRODUCTION AND PRELIMINARIES

In this part of the paper, we prove Theorem 1.4. Indeed we show that, using minmax theory for anisotropies sufficiently close to the area, one can realize p -widths with a collection of smooth hypersurfaces, thus establishing Assumption 5.1 for $3 \leq d \leq 6$. We follow the

Almgren-Pitts scheme developed for anisotropies in [18] and describe the modifications needed there-in to adapt the situation to multi-parameter minmax and establish regularity.

The first main adaptation is the sweepout; in particular we need to work with multi-parameter settings, we cannot perform the surgeries in [18], since they are one-parameter in nature, as they rely on the construction of nested sweepouts introduced in [7], and subsequently refined in [6, 12]. This means that we cannot expect an apriori bound on the density ratio and we rely on Schoen-Simon type curvature estimates for anisotropies near the area functional. For this we use the Bernstein theorems of Lin [36], Chodosh-Li [9, Corollary 1.5], and Li-Xia [33, Corollary 3] for anisotropies near the area functional and apriori smoothness for a C^3 -neighbourhood of the area in [46], with a contradiction argument.

Second, gaining the interior smoothness of replacements from the above theorem, we prove the smooth gluing of consecutive replacements using using a boundary-barrier argument inspired by [30] and the replacement ideas of [18].

Another modification is dealing with potential singularities that apriori, encode the unstable regions in the minmax varifold. In the case of the area functional we can rule them out since the monotonicity formula implies they have zero capacity, and this means their existence can be ignored in the stability inequality. Indeed here the monotonicity formula is out of reach, and instead we use the removable singularity strategy of *separation function* developed in [21] and the near area calculations of [53].

8.1. The setup. Throughout this part, (M^d, g) is closed, connected, and smooth, with $3 \leq d \leq 6$. We consider smooth integrands satisfying

$$\|F - 1\|_{C^{k_0}(G_{d-1}(M))} < \delta_0, \quad k_0 \geq 6,$$

where $\delta_0 > 0$ will be chosen sufficiently small, independently of the number of parameters. In particular, these integrands are uniformly strictly elliptic. We retain the notation and conventions of Section 2.2.

For a varifold V and a smooth vector field Y , with flow ψ_t , we write

$$\delta \mathbf{F}(V)[Y] := \left. \frac{d}{dt} \right|_{t=0} \mathbf{F}((\psi_t)_\# V).$$

We call V *F-stationary* in an open set $U \subset M$ if $\delta \mathbf{F}(V)[Y] = 0$ for every smooth vector field compactly supported in U . A smooth F -stationary hypersurface is *stable* in U if its second variation is nonnegative for every smooth normal variation compactly supported in U . For a one-sided hypersurface, variations are understood as sections of its normal bundle.

We work with the space $\mathcal{Z}_{d-1}^0(M; \mathbb{Z}_2)$ of cycles in the flat path component of zero. We denote the usual varifold metric by $\mathbb{F}$ and equip this cycle space with the metric

$$\mathbb{F}(T, T') := \mathcal{F}(T, T') + \mathbb{F}(|T|, |T'|).$$

The energy $T \mapsto \mathbf{F}(T)$ is continuous in this topology. Moreover, a $\mathbb{F}$ -continuous family on a compact parameter space has no concentration of mass: its associated mass measures form a weakly compact family of nonatomic measures.

For the min-max construction, we allow $\mathbb{F}$ -continuous families. This is consistent with the strong mass formulation used earlier: strong mass continuity implies $\mathbb{F}$ -continuity, while Proposition 2.1 approximates every such family by a mass-continuous family in the same flat homotopy class, with arbitrarily small increase in the maximal anisotropic energy.

We use the continuous formulation of the Almgren-Pitts homotopy classes in [39, Section 2].

Definition 8.1. Let X be a finite cubical subcomplex of some $I(m, j)$, and let

$$\Phi_0 : X \longrightarrow \mathcal{Z}_{d-1}^0(M; \mathbb{Z}_2)$$

be $\mathbb{F}$ -continuous. We define its homotopy class Π to consist of all $\mathbb{F}$ -continuous maps $\Phi : X \rightarrow \mathcal{Z}_{d-1}^0(M; \mathbb{Z}_2)$ which are homotopic to Φ_0 in the flat topology. Thus there is a flat-continuous map

$$\Psi : [0, 1] \times X \longrightarrow \mathcal{Z}_{d-1}^0(M; \mathbb{Z}_2)$$

such that $\Psi(0, \cdot) = \Phi_0$ and $\Psi(1, \cdot) = \Phi$.

The associated min-max value is

$$\mathbf{L}(\Pi) := \inf_{\Phi \in \Pi} \sup_{x \in X} \mathbf{F}(\Phi(x)). \quad (8.1)$$

For a sequence $S = \{\Phi_i\}_{i \in \mathbb{N}} \subset \Pi$, we set

$$\mathbf{L}(S) := \limsup_{i \rightarrow \infty} \sup_{x \in X} \mathbf{F}(\Phi_i(x)).$$

We call S a *min-max sequence* if $\mathbf{L}(S) = \mathbf{L}(\Pi)$. Such sequences exist by the definition of the infimum. We will apply the construction when

$$\mathbf{L}(\Pi) > 0.$$

Given a min-max sequence S , a sequence of slices $\{\Phi_{i_j}(x_j)\}$, with $i_j \rightarrow \infty$, is *critical* if

$$\mathbf{F}(\Phi_{i_j}(x_j)) \longrightarrow \mathbf{L}(\Pi).$$

We define the critical set $\mathbf{C}(S)$ to be the collection of varifold limits of critical sequences:

$$\mathbf{C}(S) := \{V : |\Phi_{i_j}(x_j)| \rightharpoonup V, \quad \mathbf{F}(\Phi_{i_j}(x_j)) \rightarrow \mathbf{L}(\Pi), \quad i_j \rightarrow \infty\}.$$

By comparability,

$$\limsup_{i \rightarrow \infty} \sup_{x \in X} \mathbf{M}(\Phi_i(x)) \leq \lambda^{-1} \mathbf{L}(\Pi).$$

Consequently, varifold compactness and continuity of $\mathbf{F}$ under varifold convergence imply that $\mathbf{C}(S)$ is nonempty and compact, and

$$\mathbf{F}(V) = \mathbf{L}(\Pi) \quad \text{for every } V \in \mathbf{C}(S).$$

Finally, if Φ_0 is a p -sweepout, every map in Π is also a p -sweepout, by flat homotopy invariance of $\bar{\lambda}^p$. Proposition 2.1 and (8.1) therefore give

$$\omega_p^{d-1}(M, F) \leq \mathbf{L}(\Pi) \leq \sup_{x \in X} \mathbf{F}(\Phi_0(x)). \quad (8.2)$$

8.2. Main results. We state the min-max theorem for sequences of homotopy classes whose parameter complexes lie in a fixed cube. This includes the case of a single homotopy class and allows us to deduce smooth realization of every width directly.

Theorem 8.2. *Let (M^d, g) be closed, connected, and smooth, with $3 \leq d \leq 6$, and fix $k_0 \geq 6$. There exists $\delta_0 > 0$ with the following property. Let F be a smooth positive integrand satisfying*

$$\|F - 1\|_{C^{k_0}(G_{d-1}(M))} < \delta_0,$$

and let Π_j be homotopy classes as in Definition 8.1, with parameter complexes X_j contained in subdivisions of a fixed cube I^m . Suppose that

$$\mathbf{L}(\Pi_j) \longrightarrow L \in (0, \infty).$$

Then there exist maps $\Phi_j \in \Pi_j$ satisfying

$$\sup_{x \in X_j} \mathbf{F}(\Phi_j(x)) \leq \mathbf{L}(\Pi_j) + j^{-1}$$

and, after passing to a subsequence, points $x_j \in X_j$ such that

$$|\Phi_j(x_j)| \rightharpoonup V = \sum_{i=1}^s m_i |\Sigma_i|, \quad \mathbf{F}(V) = L.$$

Here $m_i \in \mathbb{N}$, and the Σ_i are smooth, closed, embedded, connected, pairwise disjoint F -stationary hypersurfaces.

The constant δ_0 is independent of m , the homotopy classes, and their min-max values.

In particular, taking $\Pi_j = \Pi$ with $\mathbf{L}(\Pi) > 0$ gives a min-max sequence $S \subset \Pi$ and a smooth F -stationary integral varifold $V \in \mathbf{C}(S)$, satisfying $\mathbf{F}(V) = \mathbf{L}(\Pi)$. Theorem 8.2 will be proved in the remaining sections of this part. We first derive the smooth realization theorem.

Proof of Theorem 1.4. Choose δ_0 as in Theorem 8.2, and fix an integrand F in this neighborhood and $p \geq 1$. By the definition of the width, there are mass-continuous p -sweepouts

$$\Phi_{0,j} : X_j \longrightarrow \mathcal{Z}_{d-1}^0(M; \mathbb{Z}_2)$$

such that

$$\sup_{x \in X_j} \mathbf{F}(\Phi_{0,j}(x)) \leq \omega_p^{d-1}(M, F) + j^{-1}.$$

We may choose the X_j as cubical subcomplexes of subdivisions of the fixed cube I^{2p+1} , by the parameter-space reduction in [35, proof of Proposition 3.2]. Indeed, restricting a sweepout to the p -skeleton preserves the sweepout condition, since restriction is injective on $H^p(\cdot; \mathbb{Z}_2)$. After a piecewise-linear embedding of this skeleton into the interior of I^{2p+1} , take a cubical neighborhood contained in a regular neighborhood and precompose the restricted sweepout with the retraction onto the skeleton. The retraction has a section, so the pullback of $\bar{\lambda}^p$ remains nonzero. These operations preserve mass continuity and do not increase the maximal anisotropic energy.

Let Π_j be the homotopy class of $\Phi_{0,j}$. By (8.2),

$$0 < \omega_p^{d-1}(M, F) \leq \mathbf{L}(\Pi_j) \leq \sup_{x \in X_j} \mathbf{F}(\Phi_{0,j}(x)) \leq \omega_p^{d-1}(M, F) + j^{-1}.$$

Consequently,

$$\mathbf{L}(\Pi_j) \longrightarrow \omega_p^{d-1}(M, F).$$

The parameter complexes lie in the fixed cube I^{2p+1} , so Theorem 8.2 provides an integral varifold

$$V = \sum_{i=1}^s m_i |\Sigma_i|, \quad \mathbf{F}(V) = \omega_p^{d-1}(M, F),$$

where the Σ_i have all the asserted smoothness, embeddedness, and stationarity properties. Since the constant δ_0 in that theorem is independent of the parameter dimension, the same δ_0 works for every p . $\square$

Remark 8.3. In the light of the recent resolution of the Bernstein problem in dimension $n = 7$ in [31], it is perhaps possible to push the near area calculations to this dimensions as well. This seems like a task that current commercially available LLMs can achieve and are optimized to do.

9. ALMOST MINIMIZING VARIFOLDS

9.1. Pull tight. Fix the homotopy classes $\{\Pi_j\}_j$, parameter complexes $X_j \subset I^m$, and limiting value

$$\mathbf{L}(\Pi_j) \longrightarrow L \in (0, \infty)$$

from the hypotheses of Theorem 8.2. For sequences $S = \{\Phi_j\}$ with $\Phi_j \in \Pi_j$ and $\sup_{X_j} \mathbf{F}(\Phi_j) \rightarrow L$, we extend the previous notation by setting

$$\mathbf{C}(S) := \{V : |\Phi_{j_k}(x_k)| \rightharpoonup V, \quad \mathbf{F}(\Phi_{j_k}(x_k)) \rightarrow L, \quad j_k \rightarrow \infty, \quad x_k \in X_{j_k}\}.$$

Thus every $V \in \mathbf{C}(S)$ satisfies $\mathbf{F}(V) = L$.

We use the cycle formulation of almost minimizing. A cycle $T \in \mathcal{Z}_{d-1}^0(M; \mathbb{Z}_2)$ is (ε, δ) -almost minimizing for F in U if every finite sequence $T = T_0, \dots, T_q \in \mathcal{Z}_{d-1}^0(M; \mathbb{Z}_2)$ satisfying

$$\text{spt}(T_\ell - T) \subset U, \quad \mathcal{F}(T_\ell, T_{\ell-1}) \leq \delta, \quad \mathbf{F}(T_\ell) \leq \mathbf{F}(T) + \delta \quad (1 \leq \ell \leq q)$$

also satisfies

$$\mathbf{F}(T_q) \geq \mathbf{F}(T) - \varepsilon.$$

A varifold V is *almost minimizing for F in U* if there exist $\varepsilon_i, \delta_i \downarrow 0$ and $(\varepsilon_i, \delta_i)$ -almost minimizing cycles T_i in U such that $|T_i| \rightarrow V$.

Proposition 9.1. *There exist representatives $\Phi_j \in \Pi_j$ satisfying*

$$\sup_{x \in X_j} \mathbf{F}(\Phi_j(x)) \leq \mathbf{L}(\Pi_j) + j^{-1}$$

such that, writing $S = \{\Phi_j\}$, the following hold:

(i) *Every $W \in \mathbf{C}(S)$ is F -stationary:*

$$\delta \mathbf{F}(W)[Y] = 0 \quad \text{for every smooth vector field } Y.$$

(ii) *There exists $V \in \mathbf{C}(S)$ which is almost minimizing for F in small annuli: for every $x \in M$, there is $r_x > 0$ such that V is almost minimizing in*

$$B_t(x) \setminus \overline{B_s(x)} \quad \text{whenever } 0 < s < t < r_x.$$

In particular, $V \neq 0$ and $\mathbf{F}(V) = L$.

Moreover, there exist finitely many points $p_1, \dots, p_N \in M$ such that, setting

$$P := \{p_1, \dots, p_N\} \subset M,$$

the varifold V is almost minimizing in a sufficiently small ball about every point of $M \setminus P$. The ball radii can be chosen uniformly on each compact subset of $M \setminus P$.

Proof. Choose representatives $\Phi_j^0 \in \Pi_j$ with maximal energy at most $\mathbf{L}(\Pi_j) + (2j)^{-1}$. Comparability gives a uniform mass bound. Apply the pull-tight construction of [39, Section 2.8], replacing the area functional and its first variation by $\mathbf{F}$ and $\delta \mathbf{F}$, as in [16, Section 4.1]. The resulting deformation preserves the homotopy classes, does not increase $\mathbf{F}$, and makes every critical varifold F -stationary.

By [39, Proposition 3.7], we may further approximate these families uniformly in the $\mathbb{F}$ -metric by mass-continuous maps in the same classes. Choose the approximation errors tending to zero and the energy increase at most $(2j)^{-1}$. Since $\mathbf{F}$ is continuous under varifold convergence, this preserves the critical set. Denoting the resulting sequence by $S = \{\Phi_j\}$, we obtain the asserted energy bound and (i).

For (ii), use the combinatorial deformation argument of [42, Theorem 4.10]; see [35, proof of Lemma 3.2] for varying parameter complexes and [18, proof of Theorem 4.6] for the anisotropic adaptation of the existence argument. The required modification is to measure the retained pieces by their $\mathbf{F}$ -energy, use local lower semicontinuity for pieces substituted from a flat limit, and estimate the added slicing chains by λ^{-1} times their mass, as in Proposition 2.1. The ordinary mass estimates continue to control fineness, while

$$|\mathbf{F}(T) - \mathbf{F}(T')| \leq \lambda^{-1} \mathbf{M}(T - T')$$

controls the energy errors in the extensions.

The same modification applies to the localized interpolation underlying [42, Theorem 3.9]: almost minimization with respect to mass-small steps in U implies almost minimization in the stated flat-step sense in every $W \Subset U$, with possibly different witnessing cycles and thresholds. Thus mass-step almost minimization in sufficiently small annuli gives the required flat-step property, by applying this implication in slightly larger annuli.

All combinatorial constants are uniform because the X_j lie in subdivisions of the fixed cube I^m . Consequently, if no element of $\mathbf{C}(S)$ were almost minimizing in small annuli, the cited argument would produce maps $\Psi_j \in \Pi_j$ satisfying

$$\limsup_{j \rightarrow \infty} \sup_{x \in X_j} \mathbf{F}(\Psi_j(x)) < L,$$

contrary to $\mathbf{L}(\Pi_j) \rightarrow L$. Hence such a $V \in \mathbf{C}(S)$ exists; moreover, $\mathbf{F}(V) = L > 0$, so $V \neq 0$.

Finally, choose a finite subcover

$$M = \bigcup_{a=1}^N B_{r_{p_a}/2}(p_a), \quad P = \{p_1, \dots, p_N\}.$$

For every $y \in M \setminus P$, some sufficiently small ball about y is contained in an annulus $B_t(p_a) \setminus \overline{B_s(p_a)}$ with $0 < s < t < r_{p_a}$. Almost minimization passes to smaller open sets, giving the ball conclusion. Covering each compact subset of $M \setminus P$ by finitely many smaller such balls gives a uniform choice of radius there. $\square$

9.2. Curvature estimates and regularity of replacements. For the area functional, Bellettini [4] obtained extensions of the classical curvature estimates of Schoen–Simon–Yau [47] and the regularity theory of Schoen–Simon [45] using De Giorgi iteration. In the anisotropic setting, Allard [2] established a priori estimates for the oscillation of the normal under control of the first and second variations of an elliptic integrand.

We first establish interior curvature estimates and compactness for smooth stable hypersurfaces. We then construct replacements and prove their interior regularity. Boundary regularity requires quantitative control of the prescribed boundary, which we state separately.

Proposition 9.2. *After decreasing $\delta_0 > 0$, there exist $r_0 > 0$ and $C > 0$, depending only on (M, g) and k_0 , with the following properties.*

(i) *Suppose that $\|F - 1\|_{C^{k_0}} \leq \delta_0$ and that Σ is a smooth, properly immersed, two-sided, stable F -stationary hypersurface without boundary in U . Then*

$$\sup_{\Sigma \cap B_r(x)} |A_\Sigma| \leq \frac{C}{r} \tag{9.1}$$

whenever $B_{2r}(x) \Subset U$ and $0 < r \leq r_0$. The estimate is independent of the area of Σ .

(ii) *Suppose that $F_j \rightarrow F_\infty$ smoothly, with $\|F_j - 1\|_{C^{k_0}} \leq \delta_0$, and that Σ_j are smooth, properly embedded, two-sided, stable F_j -stationary hypersurfaces without boundary in U . Assume that their areas are locally uniformly bounded in U .*

After passing to a subsequence, $|\Sigma_j|$ converges to an integral varifold whose support is a smooth, embedded, stable F_∞ -stationary hypersurface. The convergence is locally smooth as graphical sheets, with positive integer multiplicities constant on connected components. The limit may be empty.

Proof. The Bernstein theorems of Lin [36], Chodosh–Li [9, Corollary 1.5], and Li–Xia [33, Corollary 3] imply that every complete, two-sided, stable immersion in $\mathbb{R}^d$, $3 \leq d \leq 6$, stationary for an autonomous integrand sufficiently C^4 -close to area is flat. (For the isotropic stable Bernstein theorem, see [24, 25, 43] in $\mathbb{R}^3$, [10, 5] in $\mathbb{R}^4$, [11] in $\mathbb{R}^5$, and [41] in $\mathbb{R}^6$.) These are the main tools used in this proof.

Choose δ_0 so that the frozen integrands below and their limits belong to this neighborhood.

If (9.1) fails, point-picking gives points y_j , numbers $\lambda_j = |A_{\Sigma_j}|(y_j) \rightarrow \infty$, and radii ρ_j such that

$$\lambda_j \rho_j \rightarrow \infty, \quad \sup_{\Sigma_j \cap B_{\rho_j}(y_j)} |A_{\Sigma_j}| \leq 2\lambda_j.$$

Rescaling normal coordinates by λ_j , the ambient metrics converge to the Euclidean metric and the integrands converge locally in C^4 to an autonomous integrand in the Bernstein neighborhood. Uniform intrinsic graphical charts and elliptic estimates give a pointed limiting immersion satisfying $|A|(0) = 1$, and $|A| \leq 2$. It is complete, two-sided, stationary, and stable: the rescaled domains exhaust intrinsic neighborhoods of arbitrarily large radius, and the equation and stability inequality pass to the limit on compact intrinsic subsets. This contradicts the Bernstein theorem. Only a selected pointed sheet is used, so no area bound enters the argument.

For (ii), (i) gives uniform curvature bounds on compact subsets. Elliptic estimates give bounds for all derivatives, and the local area bounds control the number of sheets in smaller balls. A diagonal argument therefore gives smooth graphical convergence with integer multiplicities. Embeddedness prevents transverse intersections, and the strong maximum principle implies that touching limiting sheets coincide. Thus the support is embedded and the multiplicity is constant on each component. Stationarity and stability pass to the limit. For a one-sided limiting component, stability is understood for compactly supported sections of its normal bundle. $\square$

We next record a boundary version for a quantitatively controlled prescribed boundary. Smooth embedded convergence below includes a common local graphical radius; in particular, distinct prescribed boundary sheets do not collide.

Proposition 9.3. *Assume the hypotheses of Proposition 9.2. Work in local coordinates $(x, z) \in \mathbb{R}^{d-1} \times \mathbb{R}$, and let $\Sigma_0 = \{z = u_0(x)\}$ be a smooth F -minimal hypersurface. For $D = B_r^{d-1}$ and $a > 0$, consider the lens*

$$C = \left\{ (x, z) : x \in D, \quad |z - u_0(x)| < a \frac{r^2 - |x|^2}{r} \right\}, \quad \Gamma = \{(x, u_0(x)) : x \in \partial D\}.$$

Choose the coordinates and then r sufficiently small that $\|Du_0\| + r\|D^2u_0\| \leq a$ and both open faces of C are strictly F -convex.

Let $p \in \Gamma$, and let W be an F -stationary varifold in a neighborhood U of p . Suppose that, for some $m \in \mathbb{N}$,

$$W = m|\Sigma_0| \quad \text{in } U \setminus \overline{C},$$

and that $W \llcorner C$ is integral and has smooth embedded support which is F -stable with respect to every compactly supported normal variation in $U \cap C$.

If a is sufficiently small, then near p the interior restriction of W is a finite sum, with positive integer multiplicities, of smooth hypersurfaces extending smoothly to Γ . Moreover, if a is chosen sufficiently small depending also on m , then

$$W = m|\Sigma_0| \quad \text{in a neighborhood of } p.$$

Proof. The boundary-barrier strategy is inspired by Hardt's work [30]; see also the proof of [18, Lemma 5.2]. Here we use the interior curvature estimate and the narrow-lens geometry to obtain graphicality and smooth extension to the boundary. We then use the anisotropic conormal balance and integrality to identify the multiplicity and prove smooth matching with the exterior hypersurface.

We first exclude contact with the open faces of the lens. At every point of $\partial C \setminus \Gamma$, the exterior sheet Σ_0 is absent from a sufficiently small neighborhood. Consequently, stationarity of W across the face and its strict F -convexity exclude support contact, by the anisotropic barrier maximum principle; see [17, Section 3].

Shrink U to a coordinate neighborhood for which $U \cap C$ is simply connected. Every properly embedded component of the interior support is then two-sided in $U \cap C$. Its choice of normal also orients its extension by the empty set across the open faces. Thus the interior support, extended by the empty set across the faces, is a smooth hypersurface without boundary in $U \setminus \Gamma$. It is stable there: the restriction of any compactly supported

variation to this support can be cut off inside C , since its intersection with the support stays a positive distance from the faces. The interior curvature estimate therefore gives, after shrinking U ,

$$|A|(q) \leq \frac{C}{\text{dist}(q, \Gamma)}. \quad (9.2)$$

Crucially, C is independent of the opening a .

Put $f(x, z) = z - u_0(x)$. The definition of the lens gives

$$|f(x, z)| \leq 2a \text{dist}(x, \partial D) \leq Ca \text{dist}((x, z), \Gamma).$$

At a point whose distance from Γ is s , (9.2) bounds the second derivative of f along unit-speed intrinsic geodesics by C/s . Applying Taylor's formula along geodesic segments of length $s\sqrt{a}$ yields $|\nabla_\Sigma f| \leq C\sqrt{a}$. Hence, for small a , projection onto the x -coordinates is a local diffeomorphism.

Over a smaller simply connected boundary patch of D , this projection is proper: the lens bounds prevent vertical escape, and the exclusion of face contact prevents any additional boundary. It is therefore a covering. Local finiteness of the mass makes its degree finite. Accordingly, the interior support consists of finitely many ordered graphs $z = u_\ell(x)$, with constant positive integer multiplicities q_ℓ , satisfying

$$u_\ell = u_0 \quad \text{on } \partial D, \quad |Du_\ell - Du_0| \leq C\sqrt{a}.$$

Each graph solves the uniformly elliptic scalar equation for an F -minimal graph. Boundary $C^{1,\alpha}$ estimates for the Dirichlet problem, followed by Schauder estimates, give smooth extension to Γ .

There is no additional varifold mass concentrated on Γ . Indeed, test stationarity with cutoffs of the normal displacement field from Γ , supported in shrinking tubular neighborhoods. On Γ the field vanishes and its derivative is projection onto the two-dimensional normal space. For the area integrand, its tangential divergence on every hyperplane is at least one; this remains positive when F is sufficiently close to area. Contributions off Γ tend to zero, proving $\|W\|(\Gamma) = 0$.

We finally prove the last assertion. Along Γ , let β_0 and β_ℓ denote the anisotropic conormal vectors of $\Sigma_0 \cap C$ and the interior graphs, with the same outward convention. Stationarity gives

$$m\beta_0 = \sum_\ell q_\ell \beta_\ell.$$

More explicitly, at $p \in \Gamma$ let $N_p\Gamma = (T_p\Gamma)^\perp$, let η_ℓ be the outward unit conormal, and choose unit normals ν_ℓ so that one quarter-turn R_p of this two-plane satisfies $R_p\nu_\ell = \eta_\ell$, including $\ell = 0$. In the normal representation introduced above,

$$\beta_\ell = \varphi_p(\nu_\ell)\eta_\ell - \langle D\varphi_p(\nu_\ell), \eta_\ell \rangle_g \nu_\ell = R_p D(\varphi_p|_{N_p\Gamma})(\nu_\ell).$$

Here the last derivative is the gradient in $N_p\Gamma$. These vectors lie on the rotated Wulff curve of $\varphi_p|_{N_p\Gamma}$, which is a smooth strictly convex curve enclosing the origin by ellipticity. Let L be its supporting linear functional at β_0 , normalized by $L(\beta_0) = 1$. Then $L(\beta_\ell) \leq 1$, with equality only when $\beta_\ell = \beta_0$.

Since the graphs are smooth up to the boundary, the lens height bound also gives $|Du_\ell - Du_0| \leq Ca$ on ∂D . We may therefore choose a , depending on m , so that

$$L(\beta_\ell) > \frac{m}{m+1} \quad \text{for every } \ell.$$

Writing $N = \sum_\ell q_\ell$ and applying L to the conormal identity gives

$$\frac{m}{m+1}N < m \leq N.$$

Thus $N = m$, and equality in the supporting inequality forces every $\beta_\ell = \beta_0$.

Each interior graph consequently joins Σ_0 with matching first derivatives. The joined graph is a weak solution of the graphical Euler-Lagrange equation and is smooth by elliptic regularity. Unique continuation identifies it with Σ_0 . Summing its multiplicities proves $W = m|\Sigma_0|$ near p . $\square$

Smooth replacements. Let V be an F -stationary varifold in M and let $K \subset M$ be compact. A *smooth F -replacement* of V in K is an F -stationary varifold V^* such that

$$V^* \llcorner (M \setminus K) = V \llcorner (M \setminus K), \quad \mathbf{F}(V^*) = \mathbf{F}(V),$$

and whose restriction to $\text{int } K$ is integral, with smooth embedded F -stable support and positive integer multiplicities constant on connected components. The interior restriction may be zero. Stability is understood with respect to compactly supported normal variations; the limiting support need not be two-sided. We prove below existence and regularity of replacements.

Proposition 9.4. *Assume the dimensional and near-area hypotheses of Proposition 9.2, decreasing δ_0 if necessary. Let V be F -stationary in M and almost minimizing for F in an open set U .*

- (i) *For every compact $K \Subset U$, there exists a smooth F -replacement V^* of V in K which remains almost minimizing in U . Its interior support satisfies the curvature estimate (9.1) whenever $B_{2r}(x) \Subset \text{int } K$ and $r \leq r_0$, with the same constant C .*
- (ii) *The construction can be iterated any finite number of times in compact subsets of the same open set U . At every stage, stationarity, almost minimization in U , and the total anisotropic energy are preserved. The replacement regions may have corners; in particular, they may be closed round annuli or closed adapted lens annuli.*
- (iii) *Suppose that, near an outer-edge point $p \in \Gamma$, the region K agrees with the closure of an adapted lens C from Proposition 9.3, and that*

$$V = m|\Sigma_0| \quad \text{outside } \overline{C} \quad \text{near } p, \quad m \in \mathbb{N}.$$

Then the boundary conclusions of that proposition apply to V^ . In particular, for sufficiently small opening, the interior sheets extend smoothly to Γ with integer multiplicities. If the opening is sufficiently small depending also on m , then*

$$V^* = m|\Sigma_0| \quad \text{near } p.$$

This applies at the outer edge of an adapted lens annulus, since its inner boundary is absent from a sufficiently small neighborhood of p .

The threshold δ_0 is independent of multiplicity and of the number of replacements. In item (iii), only the opening and size of the lens may depend on m .

Proof. Choose $(\varepsilon_i, \delta_i)$ -almost minimizing cycles T_i in U with $|T_i| \rightharpoonup V$. We use constrained minimization among admissible endpoints; compare the replacement construction in [18, Lemma 4.8 and Proposition 4.11].

For each i , minimize $\mathbf{F}$ among endpoints of admissible chains starting at T_i whose changes are supported in K . Compactness for mod-2 cycles and lower semicontinuity give a flat limit of a minimizing sequence. This limit is itself admissible: append it to a sufficiently late endpoint by one flat step of size less than δ_i . Thus the minimum is attained by a cycle T_i^* , and

$$\text{spt}(T_i^* - T_i) \subset K, \quad \mathbf{F}(T_i) - \varepsilon_i \leq \mathbf{F}(T_i^*) \leq \mathbf{F}(T_i). \quad (9.3)$$

Concatenating a chain from T_i to T_i^* with any admissible chain starting at T_i^* shows that T_i^* is still $(\varepsilon_i, \delta_i)$ -almost minimizing in U . Indeed, the latter chain has energy at most $\mathbf{F}(T_i^*) + \delta_i \leq \mathbf{F}(T_i) + \delta_i$, and its endpoint has energy at least $\mathbf{F}(T_i) - \varepsilon_i \geq \mathbf{F}(T_i^*) - \varepsilon_i$.

Moreover, T_i^* locally minimizes $\mathbf{F}$ in $\text{int } K$. To see this, take a sufficiently small coordinate ball $B \Subset \text{int } K$ with $\text{Vol}(B) < \delta_i$. A cycle competitor agreeing with T_i^* outside B differs

from it by the boundary of a mod-2 filling supported in B , of mass at most $\text{Vol}(B)$. An energy-decreasing competitor would therefore be an admissible additional endpoint, contradicting minimality.

Although these minimizing neighborhoods may shrink with i , stationarity and stability hold throughout $\text{int } K$. Indeed, any sufficiently short smooth variation compactly supported there can be subdivided into flat steps smaller than δ_i , while its energy stays below $\mathbf{F}(T_i) + \delta_i$. Endpoint minimality then makes $t = 0$ a local minimum along every such variation.

After decreasing δ_0 , the near-area minimizing regularity theorem [46, Theorem II.8 and Remark II.9] implies that the support of T_i^* is a C^2 embedded hypersurface in $\text{int } K$; the dimensional restriction eliminates its singular set. Indeed, in sufficiently small normal-coordinate charts the C^{k_0} bound, with $k_0 \geq 6$, places the frozen normal integrands in a fixed compact family in the C^3 topology. After decreasing the coordinate radius and δ_0 , the frozen integrands lie in the neighborhood of area within this family required by Theorem II.8. With hypersurface dimension $d - 1$, choose $t = 8 - d > 0$ in that theorem; its singular-set estimate becomes $\mathcal{H}^0(\text{sing } T_i^*) = 0$. The compact family and the resulting smallness threshold are independent of i , the minimizing radius, and the number of parameters. Remark II.9 includes both the Riemannian and mod-2 versions needed here. Elliptic bootstrapping gives smoothness. Since T_i^* bounds a set of finite perimeter, its smooth interior support is two-sided and carries multiplicity one.

The energy bounds give a uniform mass bound. Passing to a subsequence, write $|T_i^*| \rightharpoonup V^*$. Proposition 9.2(ii) gives locally smooth graphical convergence in $\text{int } K$, with positive integer multiplicities and smooth embedded stable support. Its curvature estimate passes to this limit. In particular, multiplicity in V^* is retained even when sheets cancel in the flat limit of the cycles.

The support condition and (9.3) imply that $V^* = V$ on $M \setminus K$, and $\mathbf{F}(V^*) = \mathbf{F}(V)$. The cycles T_i^* witness almost minimization of V^* in U . This implies stationarity in U : a strictly energy-decreasing flow could be discretized into admissible flat steps and would contradict almost minimization for large i . Combining this with stationarity of V outside K , using a cutoff equal to one near K , proves stationarity of V^* on all of M . This establishes (i).

Since V^* remains almost minimizing in the same U , the construction can be repeated there, proving (ii).

Finally, under the hypotheses of (iii), V^* is stationary across the lens boundary, agrees with $m|\Sigma_0|$ on its exterior, and has integral, smooth embedded stable support on its interior. These are exactly the hypotheses of Proposition 9.3. Its conclusions prove (iii). $\square$

9.3. Smooth gluing of consecutive replacements. At this point we prove the smooth gluing of consecutive replacements in overlapping annuli, and using this show the regularity of the limit varifold in almost minimizing regions.

Proposition 9.5. *Assume the dimensional and near-area hypotheses above and Proposition 9.3. Let $0 < s < \rho < t < 2\rho$, let V' be an F -stationary replacement of V in $\overline{B_{2\rho}(x)} \setminus B_\rho(x)$, and let V'' be an F -stationary replacement of V' in $\overline{B_t(x)} \setminus B_s(x)$. Assume that these replacements have smooth embedded stable interiors with integer multiplicities, and that the replacement construction can be iterated in compact regions contained in sufficiently small admissible annuli.*

Write Σ' for the support of V' in $B_{2\rho}(x) \setminus \overline{B_\rho(x)}$. If $\partial B_t(x)$ is strictly F -convex and transverse to Σ' , then for every $y \in \Sigma' \cap \partial B_t(x)$ there exists $r_y > 0$ such that

$$V'' \llcorner B_{r_y}(y) = V' \llcorner B_{r_y}(y).$$

Consequently, V'' has smooth embedded support in $B_{2\rho}(x) \setminus \overline{B_s(x)}$, with locally constant positive integer multiplicities.

Proof. Fix $y \in \Sigma' \cap \partial B_t(x)$. Choose a neighborhood U of y in which $V' = m|\Sigma_0|$, where $m \in \mathbb{N}$ and Σ_0 is a smooth embedded F -minimal hypersurface. In particular,

$$V'' = m|\Sigma_0| \quad \text{in } U \setminus \overline{B_t(x)}.$$

After passing to local coordinates and absorbing the metric into the integrand, write

$$y = 0, \quad \Sigma_0 = \{(z, u(z)) : z \in B_{r_0}^{d-1}\}, \quad u(0) = 0, \quad Du(0) = 0.$$

Shrink the neighborhood so that $\Sigma_0 \cap B_t(x) \cap U$ is connected.

Fix a small opening $a > 0$ and define

$$C_r = \left\{ (z, h) : |z| < r, \quad |h - u(z)| < a \frac{r^2 - |z|^2}{r} \right\}, \quad A_r = C_r \setminus \overline{C_{r/2}},$$

with outer edge

$$\Gamma_r = \{(z, u(z)) : |z| = r\}.$$

For each fixed a , the two outer faces are strictly F -convex when r is sufficiently small. Moreover, $\overline{A_r}$ is contained in a round annulus centered at y , with inner radius comparable to ar and outer radius comparable to r . Thus the assumed replacement construction provides a further replacement W_r of V'' in $\overline{A_r}$.

Choose a sufficiently small for Proposition 9.3 with exterior multiplicity m . By transversality, every sufficiently small Γ_r contains a point $z_r \in \Gamma_r \setminus \overline{B_t(x)}$. Near z_r , the exterior of the lens carries precisely $m|\Sigma_0|$. The lens proposition therefore gives

$$W_r = m|\Sigma_0| \quad \text{near } z_r.$$

The inner lens is disjoint from a sufficiently small neighborhood of z_r , so it does not affect this application.

The set

$$\Sigma_0 \cap A_r = \{(z, u(z)) : r/2 < |z| < r\}$$

is connected, since $d - 1 \geq 2$. Interior smoothness, unique continuation, and constancy of multiplicity propagate the preceding identity along this entire sheet: $\Sigma_0 \cap A_r$ is contained in $\text{spt } \|W_r\|$ and has multiplicity exactly m . Indeed, the set of points where this local identification holds is open and closed in $\Sigma_0 \cap A_r$.

We claim that

$$\Gamma_r \subset \text{spt } \|V''\|. \tag{9.4}$$

Every point $q \in \Gamma_r$ belongs to $\text{spt } \|W_r\|$ by the preceding continuation. If $q \notin \text{spt } \|V''\|$, then V'' vanishes in a neighborhood of q . Since $W_r = V''$ outside $\overline{A_r}$, the support of W_r in that neighborhood lies in $\overline{C_r}$. The upper face of C_r extends smoothly through q as a strictly F -convex hypersurface. It would therefore touch the stationary varifold W_r from its containing side, contrary to the anisotropic barrier maximum principle. This proves (9.4).

As r varies, the edges Γ_r cover the punctured graph Σ_0 near y . Consequently,

$$\Sigma_0 \cap U \subset \text{spt } \|V''\|$$

after shrinking U . Inside $B_t(x)$, the support of V'' is smooth and embedded. Thus it agrees locally with Σ_0 at every point of $\Sigma_0 \cap B_t(x) \cap U$. Its multiplicity along this connected sheet is therefore a fixed integer $k \geq 1$. Other components of V'' are not yet excluded.

We now identify k . Choose a new opening sufficiently small for the lens proposition with both exterior multiplicities m and k , and take one new replacement W in the corresponding lens annulus. At an outer-edge point outside $\overline{B_t(x)}$, the lens proposition and the same continuation argument give multiplicity m along $\Sigma_0 \cap A_r$. On the other hand, transversality also supplies $q_r \in \Gamma_r \cap B_t(x)$. Since q_r lies strictly inside the original annulus, interior regularity gives

$$V'' = k|\Sigma_0| \quad \text{in a whole neighborhood of } q_r.$$

Applying the lens proposition at q_r therefore gives $W = k|\Sigma_0|$ near q_r . Comparing the two identifications on the same connected sheet $\Sigma_0 \cap A_r$ proves $k = m$.

Finally, in U the difference

$$R := V'' - m|\Sigma_0|$$

is a nonnegative F -stationary varifold. Nonnegativity follows from the multiplicity just established inside $B_t(x)$ and the exterior identity; the transverse intersection $\Sigma_0 \cap \partial B_t(x)$ carries no $(d-1)$ -dimensional mass of $|\Sigma_0|$. Moreover,

$$\text{spt } \|R\| \cap U \subset \overline{B_t(x)}.$$

Strict F -convexity of $\partial B_t(x)$ and the barrier maximum principle exclude y from $\text{spt } \|R\|$. Hence R vanishes in a neighborhood of y , proving

$$V'' = m|\Sigma_0| = V' \quad \text{near } y.$$

At points of $\partial B_t(x)$ outside $\text{spt } \|V'\|$, the same barrier principle excludes support of V'' . Together with interior regularity, this proves the final assertion. $\square$

We now conclude the regularity away from finitely many points:

Corollary 9.6. *Under the dimensional and near-area assumptions above, let V be the F -stationary varifold given by Proposition 9.1. There is a finite set $P \subset M$ such that*

$$V \llcorner (M \setminus P) = \theta |\Sigma|,$$

where $\Sigma \subset M \setminus P$ is a smooth, properly embedded F -stationary hypersurface and θ is a locally constant positive integer-valued function on Σ . For every $p \in P$, there is $\rho_p > 0$ such that $B_{\rho_p}(p) \cap P = \{p\}$ and Σ is stable in $B_{\rho_p}(p) \setminus \{p\}$.

Proof. Let P be the exceptional set supplied by Proposition 9.1. The integrality argument of [16, Lemma 4.15] applies using Proposition 9.4, the barrier principle, and Proposition 9.2. Hence V is integral.

Fix $x \in M \setminus P$ and a neighborhood U of x in which V is almost minimizing. Successive annular replacements and Proposition 9.5, followed by the identification argument of [16, proof of Proposition 4.14, Step 2] with $c = 0$, show that V has smooth properly embedded support in a punctured ball about x . Since almost minimization holds throughout U , the construction admits a uniform positive outer radius for centers near x . Repeating it at a nearby center therefore gives regularity at x as well. Thus the support is smooth and properly embedded in $M \setminus P$, and integrality and the constancy theorem give the asserted locally constant positive integer multiplicities.

Finally, choose $\rho_p > 0$ so that $B_{\rho_p}(p) \cap P = \{p\}$ and V is almost minimizing in every annulus centered at p contained in this ball. On its smooth support, almost minimization excludes negative second variation by the deformation argument of [16, proof of Proposition 4.14, Step 3, Case (a)], with $c = 0$. Every compactly supported variation in $B_{\rho_p}(p) \setminus \{p\}$ is supported in one such annulus, proving stability there. $\square$

10. REMOVING SINGULARITIES

We adapt the separation argument of [21, Section 6] to near-area integrands in dimensions $3 \leq d \leq 6$. In particular we will look at the distance of nearby sheets, called *the separation function*, and show that its difference with $\kappa|x|$ for small enough κ , is a supersolution to the JACOBI operator. Since the hypersurface is stable away from the origin, the Jacobi operator satisfies the maximum principle and this shows that the separation function cannot vanish faster than linear. This is indeed a density bound at the origin.

Theorem 10.1. *Let $3 \leq d \leq 6$ and $k_0 \geq 6$. There exists $\delta_* > 0$ such that the following holds. Let F be a smooth positive elliptic integrand on $G_{d-1}(B_1)$, where $B_1 \subset \mathbb{R}^d$, satisfying*

$$\|F - 1\|_{C^{k_0}(G_{d-1}(B_1))} < \delta_*.$$

Suppose that $\Sigma \subset B_1 \setminus \{0\}$ is a smooth, properly embedded, two-sided hypersurface without boundary which is F -stationary and F -stable away from the origin, and satisfies $\mathcal{H}^{d-1}(\Sigma) < \infty$. Then $\bar{\Sigma} \cap B_1$ is a smooth embedded, F -stationary and F -stable hypersurface.

Proof. Write $n = d - 1$, $r(x) = |x|$, and $d\mu_\Sigma = d\mathcal{H}^n \llcorner \Sigma$. By Proposition 9.2,

$$|A_\Sigma(x)| \leq \frac{K}{r(x)}$$

near the origin, with K uniform in a fixed sufficiently small neighborhood of the area integrand. With our sign convention,

$$J_F = -\operatorname{div}_\Sigma(A_F \nabla_\Sigma) + c_F,$$

and stability means

$$Q_F(\varphi, \varphi) := \int_\Sigma \varphi J_F \varphi d\mu_\Sigma = \int_\Sigma (\langle A_F \nabla_\Sigma \varphi, \nabla_\Sigma \varphi \rangle + c_F \varphi^2) d\mu_\Sigma \geq 0$$

for every $\varphi \in C_c^\infty(\Sigma)$.

Expanding the stationarity equation and the Jacobi coefficients around the area integrand gives

$$\begin{aligned} |H| &\leq C\delta_*(1 + |A_\Sigma|), & |A_F - \operatorname{Id}| &\leq C\delta_*, \\ |\nabla_\Sigma A_F| &\leq C\delta_*(1 + |A_\Sigma|), & |c_F + |A_\Sigma|^2| &\leq C\delta_*(1 + |A_\Sigma|^2). \end{aligned}$$

The formula for $\Delta_\Sigma r$ therefore yields

$$J_F r \leq -\frac{n - |\nabla_\Sigma r|^2}{r} - r|A_\Sigma|^2 + C\delta_* \left(\frac{1}{r} + 1 + |A_\Sigma| + r|A_\Sigma|^2 \right).$$

Since $|\nabla_\Sigma r| \leq 1$ and $|A_\Sigma| \leq K/r$, decreasing δ_* and then choosing $\rho > 0$ small gives

$$J_F r \leq -\frac{b}{r} \quad \text{on } \Sigma \cap B_\rho$$

for some $b > 0$.

The curvature estimate and interior elliptic estimates give graphical neighborhoods of radius $\theta r(x)$, with small slope and uniform rescaled derivative bounds. Following the separation construction of [21, Propositions 6.7-6.8], choose $\kappa \ll \theta$ and let s be the least positive normal height of a neighboring sheet, allowing neighbors on either side, wherever this height is less than κr . Denote this open set by $U \subset \Sigma \cap B_\rho$.

For completeness, the estimates needed from this construction are independent of the dimension. A neighboring normal graph $u > 0$ satisfies a uniformly elliptic linear equation obtained by subtracting the two graphical stationarity equations. Harnack and Schauder estimates on smaller neighborhoods give

$$|\nabla_\Sigma u| \leq C \frac{u}{r}, \quad |\nabla_\Sigma^2 u| \leq C \frac{u}{r^2}.$$

Taylor expansion of the normal-graph equation, whose linearization at zero is J_F , consequently gives

$$|J_F u| \leq C \left(\frac{u^2}{r^3} + \frac{u}{r} \right).$$

Proper embeddedness makes the relevant family of neighboring sheets locally finite. Taking its minimum, and using the weak minimum principle for J_F , gives

$$s = \kappa r \text{ on } \partial_\Sigma U \cap B_\rho, \quad |\nabla_\Sigma s| \leq C \frac{s}{r}, \quad J_F s \geq -C \left(\frac{s^2}{r^3} + \frac{s}{r} \right) \text{ weakly on } U.$$

Here κ is chosen sufficiently small that neighboring graphs continue across the smaller graphical neighborhoods while their heights remain below κr .

Proper embeddedness gives positive separation on a compact outer annulus. Choose $0 < a < \kappa$ so that $s > ar$ near $U \cap \partial B_\rho$. After decreasing a and ρ , on $\{s < ar\}$ we have

$$J_F(ar - s) \leq -\frac{ab}{r} + C \left(\frac{a^2}{r} + a \right) \leq -\frac{ab}{2r}.$$

Set $w = (ar - s)_+$ on U and extend it by zero. Then $0 \leq w \leq ar$, and w vanishes near the outer boundary.

Let χ_ε vanish on B_ε , equal one outside $B_{2\varepsilon}$, and satisfy $|\nabla_\Sigma \chi_\varepsilon| \leq C/\varepsilon$. The weak positive-part inequality and stability give

$$\begin{aligned} 0 \leq Q_F(\chi_\varepsilon w, \chi_\varepsilon w) &\leq -\frac{ab}{2} \int_\Sigma \chi_\varepsilon^2 \frac{w}{r} d\mu_\Sigma + \int_\Sigma w^2 \langle A_F \nabla_\Sigma \chi_\varepsilon, \nabla_\Sigma \chi_\varepsilon \rangle d\mu_\Sigma \\ &\leq -\frac{ab}{2} \int_\Sigma \chi_\varepsilon^2 \frac{w}{r} d\mu_\Sigma + Ca^2 \mu_\Sigma(B_{2\varepsilon} \setminus B_\varepsilon). \end{aligned}$$

Finite area makes the last term tend to zero. Thus $w = 0$, proving

$$s(x) \geq a|x|.$$

Together with the graphical neighborhoods, this gives $\eta_0 > 0$ such that $\Sigma \cap B_{\eta_0|x|}(x)$ contains only one graphical sheet, with uniformly bounded slope. Covering each dyadic annulus by a bounded number of such balls and summing gives

$$\mu_\Sigma(B_t(0)) \leq Ct^n.$$

For balls centered on Σ , use a graphical neighborhood when $t \ll |y|$, and inclusion in $B_{Ct}(0)$ otherwise. Enlarging intersecting balls also covers arbitrary centers. Hence, in a smaller neighborhood of the origin,

$$\mu_\Sigma(B_t(y)) \leq Ct^n.$$

The constant may depend on Σ through a .

This bound gives cutoffs around the puncture with Dirichlet energy tending to zero, using logarithmic cutoffs when $n = 2$. The near-area stability inequality also gives

$$\int_\Sigma |A_\Sigma|^2 \varphi^2 d\mu_\Sigma \leq C \int_\Sigma (|\nabla_\Sigma \varphi|^2 + \varphi^2) d\mu_\Sigma.$$

Applying this to an outer cutoff multiplied by the cutoffs around the puncture shows that $|A_\Sigma|^2$ is locally integrable across the origin. The usual cutoff arguments therefore extend stationarity and stability across the puncture.

If $n = 2$, we have in particular

$$\int_{\Sigma \cap B_{\rho/2}} |A_\Sigma|^2 d\mu_\Sigma < \infty.$$

White's removable-singularity theorem applies as in [18, proof of Theorem 5.1, Step 3], and gives the conclusion.

Suppose now that $3 \leq n \leq 5$. Fix an arbitrary sequence $r_j \downarrow 0$ and put $\Sigma_j = r_j^{-1}\Sigma$. The rescaled integrands $F(r_j x, T)$ converge smoothly on compact sets to the autonomous integrand $F_0(T) := F(0, T)$. By the curvature and density estimates and Proposition 9.2, a subsequence converges smoothly with integer multiplicities on compact subsets of $\mathbb{R}^{n+1} \setminus \{0\}$ to an F_0 -stable hypersurface Σ_∞ , possibly empty. Its support is properly embedded and two-sided in the punctured ambient space. Writing $d\mu_\infty = d\mathcal{H}^n \llcorner \Sigma_\infty$, we have

$$\mu_\infty(B_R(0)) \leq CR^n, \quad |A_{\Sigma_\infty}(x)| \leq K|x|^{-1}, \quad R > 0.$$

We spell out the metric and measure in the curvature calculation on Σ_∞ . Let g_∞ be its induced Euclidean metric, ν a unit normal, and $S = -d\nu$ its Weingarten endomorphism. Define

$$\phi_0(\xi) = |\xi| F_0(\xi^\perp), \quad B_0 = D_\xi^2 \phi_0(\nu)|_{T\Sigma_\infty}.$$

Ellipticity makes B_0 positive definite. The auxiliary metric and its volume measure are

$$g_{F_0}(X, Y) = g_\infty(B_0^{-1}X, Y), \quad d\mu_{F_0} = (\det B_0)^{-1/2} d\mu_\infty.$$

In particular, $d\mu_{F_0}$ is distinct from the anisotropic energy measure $F_0(\nu^\perp) d\mu_\infty$.

Write ∇_{F_0} and Δ_{F_0} for the gradient and Laplace-Beltrami operator of g_{F_0} . Explicitly,

$$\begin{aligned} \nabla_{F_0} f &= B_0 \nabla_{\Sigma_\infty} f, \quad \langle \nabla_{F_0} f, \nabla_{F_0} h \rangle_{F_0} = g_\infty(B_0 \nabla_{\Sigma_\infty} f, \nabla_{\Sigma_\infty} h), \\ |\nabla_{F_0} f|_{F_0}^2 &= g_\infty(B_0 \nabla_{\Sigma_\infty} f, \nabla_{\Sigma_\infty} f). \end{aligned}$$

Define the anisotropic Weingarten endomorphism and its norm by

$$S_{F_0} = B_0 S, \quad v = |S_{F_0}|_{F_0}.$$

Since S_{F_0} is self-adjoint for g_{F_0} ,

$$v^2 = \sum_{\alpha=1}^n g_{F_0}(S_{F_0} e_\alpha, S_{F_0} e_\alpha) = \text{tr}(B_0 S B_0 S),$$

where $\{e_\alpha\}$ is g_{F_0} -orthonormal. All these metrics, measures, and curvature norms are uniformly equivalent to their Euclidean counterparts when δ_* is small.

By [53, equations (10),(11)], the anisotropic curvature $v = |S_{F_0}|_{F_0}$ satisfies

$$\int_{\Sigma_\infty} v^2 f^2 d\mu_{F_0} \leq \sigma \int_{\Sigma_\infty} |\nabla_{F_0} f|_{F_0}^2 d\mu_{F_0}, \quad \frac{1}{2} \Delta_{F_0}(v^2) \geq (1 + \gamma) |\nabla_{F_0} v|_{F_0}^2 - \beta v^4,$$

where σ, β, γ can be chosen arbitrarily close to 1, 1, $2/n$, respectively, by taking F_0 sufficiently close to area.

We use these inequalities to obtain the critical exponent n , including the cases $n = 3, 4$ not covered by the exponent range in [53, equation (3)]. Set

$$q = \frac{n-4}{2}, \quad v_\varepsilon = (v^2 + \varepsilon^2)^{1/2}.$$

For a smooth cutoff η supported away from the origin, test the regularized Simons inequality with $v_\varepsilon^{2q} \eta^2$ and the stability inequality with $v_\varepsilon^{q+1} \eta$. Integration by parts and Young's inequality give

$$\int_{\Sigma_\infty} v_\varepsilon^{2q+4} \eta^2 d\mu_{F_0} \leq C \int_{\Sigma_\infty} v_\varepsilon^{2q+2} |\nabla_{F_0} \eta|_{F_0}^2 d\mu_{F_0} + C \varepsilon^2 \int_{\Sigma_\infty} v_\varepsilon^{2q+2} \eta^2 d\mu_{F_0}.$$

Indeed, the coefficient needed for absorption is

$$2q + 1 + \gamma - \beta \sigma (q + 1)^2,$$

which is positive for sufficiently small δ_* because its area value is

$$\frac{2}{n} - \frac{(n-4)^2}{4} > 0, \quad 3 \leq n \leq 5.$$

The regularization justifies the calculation also when $q = -1/2$.

Letting $\varepsilon \downarrow 0$, taking $\eta = \psi^{n/2}$, and applying Young's inequality once more yields

$$\int_{\Sigma_\infty} v^n \psi^n d\mu_{F_0} \leq C \int_{\Sigma_\infty} |\nabla_{F_0} \psi|_{F_0}^n d\mu_{F_0}.$$

Uniform equivalence of the metrics, curvature norms, and measures therefore gives

$$\int_{\Sigma_\infty} |A_{\Sigma_\infty}|^n \psi^n d\mu_\infty \leq C \int_{\Sigma_\infty} |\nabla_{\Sigma_\infty} \psi|^n d\mu_\infty. \quad (10.1)$$

For $R > 2$, choose a radial cutoff ψ_R supported where $R^{-2} < |x| < R^2$, equal to one where $R^{-1} \leq |x| \leq R$, and satisfying

$$|\nabla_{\Sigma_\infty} \psi_R| \leq \frac{C}{|x| \log R}.$$

The area-growth bound, summed over dyadic annuli, gives

$$\int_{\Sigma_\infty} |\nabla_{\Sigma_\infty} \psi_R|^n d\mu_\infty \leq C(\log R)^{1-n} \longrightarrow 0.$$

Equation (10.1) and Fatou's lemma imply $A_{\Sigma_\infty} = 0$ away from the origin.

Writing $d\mu_j = d\mathcal{H}^n \llcorner \Sigma_j$, smooth graphical convergence consequently gives, for every fixed $\tau \in (0, 1)$,

$$\int_{\Sigma_j \cap (B_1 \setminus B_\tau)} |A_{\Sigma_j}|^2 d\mu_j \longrightarrow 0.$$

On the inner ball, the rescaled curvature and density bounds instead give

$$\int_{\Sigma_j \cap B_\tau} |A_{\Sigma_j}|^2 d\mu_j \leq CK^2 \tau^{n-2}.$$

Letting first $j \rightarrow \infty$ and then $\tau \downarrow 0$, and using that the initial sequence of radii was arbitrary, we conclude

$$r^{2-n} \int_{\Sigma \cap B_r} |A_\Sigma|^2 d\mu_\Sigma \longrightarrow 0 \quad \text{as } r \downarrow 0.$$

Apply the regularity criterion in [18, proof of Theorem A.1, equation (A.4)] to the constant sequence Σ . The mass ratios are uniformly bounded, stationarity and stability extend across the puncture, and $\mathcal{H}^{n-2}(\{0\}) = 0$. The preceding limit excludes the origin from the curvature-concentration set, giving smooth extension.

Stationarity and stability persist on the extension. The ordering of the graphical sheets and the strong maximum principle give embeddedness. $\square$

Now we are ready to prove Theorem 8.2 and conclude with the section.

Proof of Theorem 8.2. Choose $\delta_0 > 0$ sufficiently small for the preceding local regularity results. Proposition 9.1 gives representatives $\Phi_j \in \Pi_j$ satisfying

$$\sup_{x \in X_j} \mathbf{F}(\Phi_j(x)) \leq \mathbf{L}(\Pi_j) + j^{-1}$$

and an F -stationary varifold $V \in \mathbf{C}(\{\Phi_j\})$ with $\mathbf{F}(V) = L > 0$. By the definition of the critical set, after passing to a subsequence there are $x_j \in X_j$ such that $|\Phi_j(x_j)| \rightarrow V$.

By Corollary 9.6 and its proof, V is integral and there is a finite set $P \subset M$ such that

$$V \llcorner (M \setminus P) = \theta |\Sigma|, \quad \mathcal{H}^{d-1}(\Sigma) \leq \|V\|(M) < \infty,$$

where Σ is smooth, properly embedded and F -stationary, and θ is locally constant with positive integer values. Moreover, Σ is stable in a sufficiently small punctured ball about each $p \in P$.

Fix $p \in P$. Since $d \geq 3$, this punctured coordinate ball is simply connected, so proper embeddedness implies that Σ is two-sided there. In orthonormal normal coordinates, rescaled by $y \mapsto \exp_p(r y)$, absorb the metric area Jacobian into the integrand and divide the functional by r^{d-1} . The resulting Euclidean integrand satisfies

$$\|F_{p,r} - 1\|_{C^{k_0}(G_{d-1}(B_1))} \leq C\delta_0 + o(1) \quad \text{as } r \downarrow 0,$$

uniformly in p . After decreasing δ_0 and then choosing r small, Theorem 10.1 therefore applies and extends Σ smoothly across p , together with its integer multiplicity.

Applying this at every point of P gives a smooth closed embedded F -stationary hypersurface. Since V is integral, $\|V\|(P) = 0$, so the extended hypersurface with its multiplicities represents the whole varifold. Its compactness gives finitely many connected components, and hence

$$V = \sum_{i=1}^s m_i |\Sigma_i|, \quad m_i \in \mathbb{N}, \quad \mathbf{F}(V) = L,$$

with the asserted properties. All smallness requirements on δ_0 come from the local analytic results and the coordinate estimates, and are independent of the parameter dimension, the homotopy classes, and their min-max values. $\square$

REFERENCES

- [1] ALLARD, W. K. A characterization of the area integrand. In *Symposia Mathematica, Vol. XIV*. 1974, pp. 429–444. [4](#)
- [2] ALLARD, W. K. An a priori estimate for the oscillation of the normal to a hypersurface whose first and second variation with respect to an elliptic integrand is controlled. *Invent. Math.* **73**, 2 (1983), 287–331. [45](#)
- [3] ALMGREN, JR., F. J. The homotopy groups of the integral cycle groups. *Topology* **1** (1962), 257–299. [2](#), [9](#), [24](#)
- [4] BELLETTINI, C. Extensions of Schoen–Simon–Yau and Schoen–Simon theorems via iteration à la De Giorgi. *Inventiones Mathematicae* **240**, 1 (2025), 1–34. [45](#)
- [5] CATINO, G., MASTROLIA, P., AND RONCORONI, A. Two rigidity results for stable minimal hypersurfaces. *Geometric and Functional Analysis* **34**, 1 (2024), 1–18. [45](#)
- [6] CHAMBERS, G. R., AND LIKUMOVICH, Y. Existence of minimal hypersurfaces in complete manifolds of finite volume. *Invent. Math.* **219**, 1 (2020), 179–217. [41](#)
- [7] CHAMBERS, G. R., AND ROTMAN, R. Monotone homotopies and contracting discs on riemannian surfaces. *Journal of Topology and Analysis* **10**, 2 (2018), 323–354. [41](#)
- [8] CHEN, L., AND LAN, J. Multiplicity one for strictly stable anisotropic allen-cahn minimal hypersurfaces, 2026. [4](#)
- [9] CHODOSH, O., AND LI, C. Stable anisotropic minimal hypersurfaces in $\mathbf{R}^4$. *Forum Math. Pi* **11** (2023), e3. [41](#), [45](#)
- [10] CHODOSH, O., AND LI, C. Stable minimal hypersurfaces in $\mathbf{R}^4$. *Acta Mathematica* **233**, 1 (2024), 1–31. [45](#)
- [11] CHODOSH, O., LI, C., MINTER, P., AND STRYKER, D. Stable minimal hypersurfaces in $\mathbf{R}^5$. *Annals of Mathematics* **204**, 1 (2026), 423–453. [45](#)
- [12] CHODOSH, O., LIKUMOVICH, Y., AND SPOLAOR, L. Singular behavior and generic regularity of min-max minimal hypersurfaces. *Ars Inven. Anal.* (2022), Paper No. 2, 27. [41](#)
- [13] CHODOSH, O., AND MANTOULIDIS, C. Minimal surfaces and the Allen–Cahn equation on 3-manifolds: index, multiplicity, and curvature estimates. *Annals of Mathematics* **191**, 1 (2020), 213–328. [2](#)
- [14] CHODOSH, O., AND MANTOULIDIS, C. The p -widths of a surface. *Publications Mathématiques de l’IHÉS* **137** (2023), 245–342. [2](#)
- [15] DANESHPAJOUH, H. R., KARASEV, R., AND VOLOVIKOV, A. Y. Hedetniemi’s conjecture from the topological viewpoint. *Journal of Combinatorial Theory, Series A* **195** (2023), 105721. [24](#)
- [16] DE PHILIPPIS, G., AND DE ROSA, A. The anisotropic min-max theory: Existence of anisotropic minimal and CMC surfaces. *Comm. Pure Appl. Math.* **77**, 7 (2024), 3184–3226. [4](#), [44](#), [51](#)
- [17] DE PHILIPPIS, G., DE ROSA, A., AND HIRSCH, J. The area blow up set for bounded mean curvature submanifolds with respect to elliptic surface energy functionals. *Discrete Contin. Dyn. Syst.* **39**, 12 (2019), 7031–7056. [46](#)
- [18] DE PHILIPPIS, G., DE ROSA, A., AND LI, Y. Existence and regularity of min-max anisotropic minimal hypersurfaces, 2024. Preprint, arXiv:2409.15232v1. [4](#), [41](#), [44](#), [46](#), [48](#), [53](#), [55](#)
- [19] DE ROSA, A. On the theory of anisotropic minimal surfaces. *Notices Amer. Math. Soc.* **71**, 7 (2024), 853–859. [4](#)
- [20] DE ROSA, A. Lectures on existence and regularity of anisotropic minimal surfaces. Preprint, arXiv:2608.10275. [4](#)
- [21] DE ROSA, A., HALAVATI, A., AND WANG, L. Min-max construction of anisotropic minimal surfaces with genus bound, 2026. Preprint, arXiv:2606.30098. [4](#), [6](#), [41](#), [51](#), [52](#)
- [22] DE ROSA, A., AND PIGATI, A. Anisotropic min-max via phase transitions. Preprint, arXiv:2512.07039. [4](#)
- [23] DEY, A. A sub-additive inequality for the volume spectrum. *Journal of Topology and Analysis* **17**, 5 (2025), 1543–1561. [22](#)
- [24] DO CARMO, M., AND PENG, C. Stable complete minimal surfaces in $\mathbb{R}^3$ are planes. *Bulletin of the American Mathematical Society (N.S.)* **1**, 6 (1979), 903–906. [45](#)
- [25] FISCHER-COLBRIE, D., AND SCHOEN, R. The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature. *Communications on Pure and Applied Mathematics* **33**, 2 (1980), 199–211. [45](#)

- [26] GIVENTAL, A. B. Nonlinear generalization of the Maslov index. In *Theory of Singularities and its Applications*, vol. 1 of *Advances in Soviet Mathematics*. American Mathematical Society, Providence, RI, 1990, pp. 71–103. [24](#)
- [27] GRANJA, G., KARSHON, Y., PABINIAK, M., AND SANDON, S. Givental’s non-linear Maslov index on lens spaces. *International Mathematics Research Notices* 2021, 23 (2021), 18225–18299. [24](#)
- [28] GROMOV, M. Dimension, non-linear spectra and width. In *Geometric Aspects of Functional Analysis*, J. Lindenstrauss and V. D. Milman, Eds., vol. 1317 of *Lecture Notes in Mathematics*. Springer, Berlin, Heidelberg, 1988, pp. 132–184. [2](#), [11](#)
- [29] GUTH, L. Minimax problems related to cup powers and Steenrod squares. *Geom. Funct. Anal.* 18, 6 (2009), 1917–1987. [2](#), [11](#)
- [30] HARDT, R. On boundary regularity for integral currents or flat chains modulo two minimizing the integral of an elliptic integrand. *Communications in Partial Differential Equations* 2, 12 (1977), 1163–1232. [41](#), [46](#)
- [31] HONG, H., LI, H., AND WANG, G. The stable Bernstein theorem in $\mathbb{R}^7$. Preprint, arXiv:2609.15720 [math.DG] (2026), 2026. [43](#)
- [32] IRIE, K., MARQUES, F. C., AND NEVES, A. Density of minimal hypersurfaces for generic metrics. *Ann. of Math. (2)* 187, 3 (2018), 963–972. [2](#)
- [33] LI, J., AND XIA, C. Stable anisotropic minimal hypersurfaces in $\mathbb{R}^5$ and $\mathbb{R}^6$. *J. Geom. Anal.* 36 (2026), 312. Article 312; preprint version arXiv:2505.16595v2. [41](#), [45](#)
- [34] LI, M. M.-C., AND ZHOU, X. Min-max theory for free boundary minimal hypersurfaces I: Regularity theory. *J. Differential Geom.* 118, 3 (2021), 487–553. [8](#)
- [35] LI, Y. Existence of infinitely many minimal hypersurfaces in higher-dimensional closed manifolds with generic metrics. *J. Differential Geom.* 124, 2 (2023), 381–395. Cited theorem numbering refers to arXiv:1901.08440v2. [2](#), [43](#), [44](#)
- [36] LIN, F.-H. Estimates for surfaces which are stationary for an elliptic parametric integral. *J. Partial Differential Equations* 3, 3 (1990), 78–92. [41](#), [45](#)
- [37] LIOKUMOVICH, Y., MARQUES, F. C., AND NEVES, A. Weyl law for the volume spectrum. *Ann. of Math. (2)* 187, 3 (2018), 933–961. Cited theorem numbering refers to arXiv:1607.08721v2. [2](#), [5](#), [8](#), [9](#), [10](#), [11](#), [12](#), [13](#), [14](#), [24](#)
- [38] MARQUES, F. C., AND NEVES, A. Existence of infinitely many minimal hypersurfaces in positive Ricci curvature. *Invent. Math.* 209, 2 (2017), 577–616. [2](#)
- [39] MARQUES, F. C., AND NEVES, A. Morse index of multiplicity one min-max minimal hypersurfaces. *Advances in Mathematics* 378 (2021), 107527. [24](#), [41](#), [44](#)
- [40] MARQUES, F. C., NEVES, A., AND SONG, A. Equidistribution of minimal hypersurfaces for generic metrics. *Invent. Math.* 216, 2 (2019), 421–443. Cited theorem numbering refers to arXiv:1712.06238v2. [2](#), [3](#), [4](#), [29](#)
- [41] MAZET, L. Stable minimal hypersurfaces in $\mathbb{R}^6$, 2024. [45](#)
- [42] PITTS, J. T. *Existence and Regularity of Minimal Surfaces on Riemannian Manifolds*, vol. 27 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1981. [2](#), [44](#)
- [43] POGORELOV, A. V. On the stability of minimal surfaces. *Doklady Akademii Nauk SSSR* 260, 2 (1981), 293–295. English translation: *Soviet Mathematics Doklady* 24 (1981), 274–276. [45](#)
- [44] RESHETNYAK, Y. G. Weak convergence of completely additive vector functions on a set. *Siberian Mathematical Journal* 9, 6 (1968), 1039–1045. [10](#)
- [45] SCHOEN, R., AND SIMON, L. Regularity of stable minimal hypersurfaces. *Comm. Pure Appl. Math.* 34, 6 (1981), 741–797. [2](#), [45](#)
- [46] SCHOEN, R., SIMON, L., AND ALMGREN, JR., F. J. Regularity and singularity estimates on hypersurfaces minimizing parametric elliptic variational integrals. *Acta Math.* 139 (1977), 217–265. [4](#), [41](#), [49](#)
- [47] SCHOEN, R., SIMON, L., AND YAU, S.-T. Curvature estimates for minimal hypersurfaces. *Acta Mathematica* 134, 3–4 (1975), 275–288. [45](#)
- [48] SMALE, S. An infinite dimensional version of Sard’s theorem. *Amer. J. Math.* 87, 4 (1965), 861–866. [36](#)
- [49] SONG, A. Existence of infinitely many minimal hypersurfaces in closed manifolds. *Ann. of Math. (2)* 197, 3 (2023), 859–895. [2](#)
- [50] SONG, A., AND ZHOU, X. Generic scarring for minimal hypersurfaces along stable hypersurfaces. *Geom. Funct. Anal.* 31, 4 (2021), 948–980. [2](#)
- [51] WHITE, B. The space of minimal submanifolds for varying Riemannian metrics. *Indiana Univ. Math. J.* 40, 1 (1991), 161–200. [2](#), [33](#), [34](#)
- [52] WHITE, B. On the bumpy metrics theorem for minimal submanifolds. *American Journal of Mathematics* 139, 4 (2017), 1149–1155. [2](#)
- [53] WINKLMANN, S. Pointwise curvature estimates for F -stable hypersurfaces. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 22, 5 (2005), 543–555. [41](#), [54](#)

- [54] YAU, S.-T. Problem section. In *Seminar on Differential Geometry*, S.-T. Yau, Ed., vol. 102 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1982, pp. 669–706. [2](#)
- [55] ZHOU, X. On the multiplicity one conjecture in min-max theory. *Ann. of Math. (2)* 192, 3 (2020), 767–820. Cited theorem numbering refers to arXiv:1901.01173v2. [2](#)
- [56] ZHOU, X., AND ZHU, J. J. Min-max theory for constant mean curvature hypersurfaces. *Invent. Math.* 218, 2 (2019), 441–490. [4](#)

DEPARTMENT OF DECISION SCIENCES AND BIDSa, BOCCONI UNIVERSITY, MILANO, ITALY
Email address: `antonio.derosa@unibocconi.it`

DEPARTMENT OF DECISION SCIENCES AND BIDSa, BOCCONI UNIVERSITY, MILANO, ITALY
Email address: `aria.halavati@unibocconi.it`