The magnetic Ginzburg-Landau equation: Stability, Regularity and Rigidity

by

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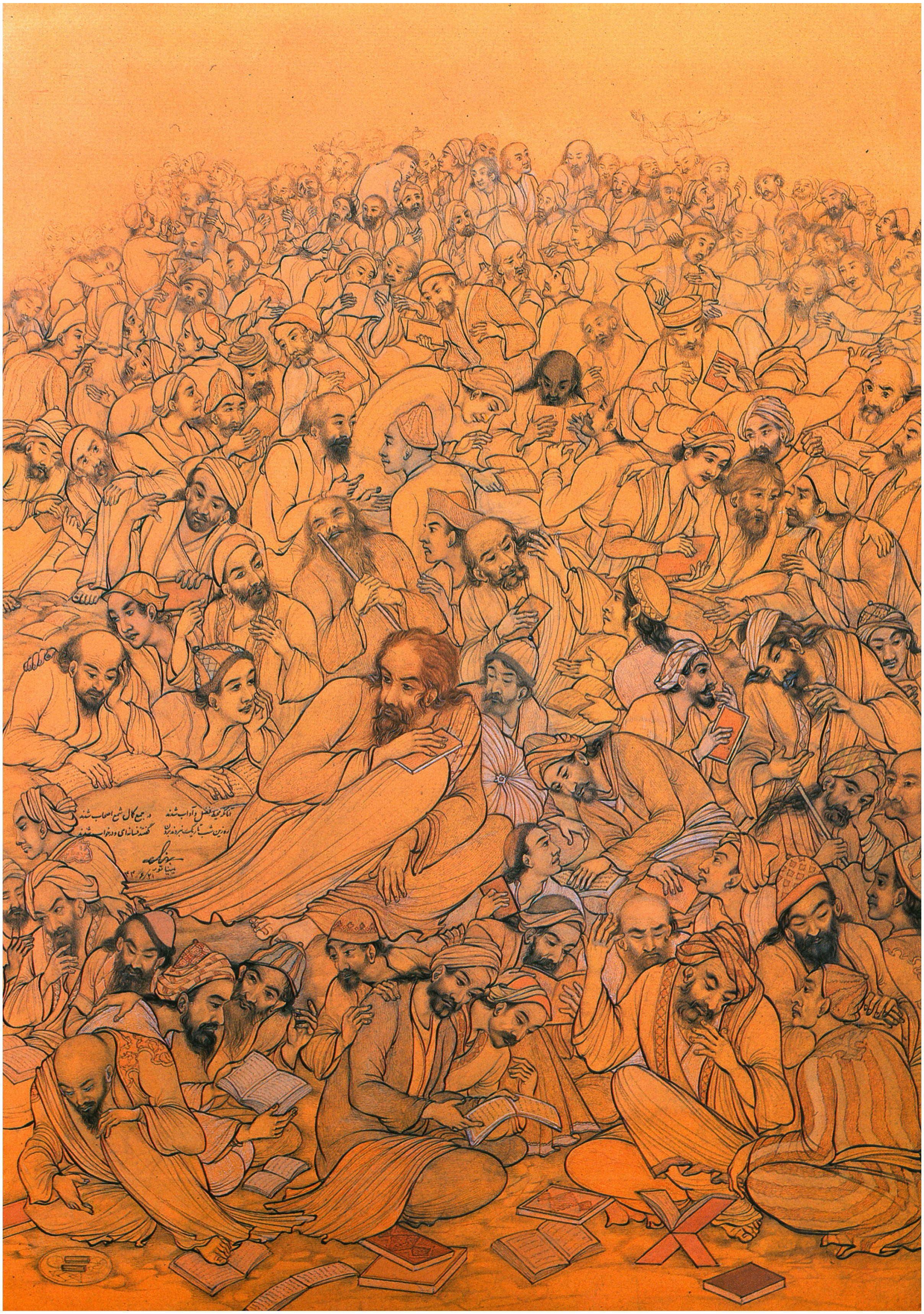

آنانکه محیط فضل و آداب شدند
در جمع کمال شمع اصحاب شدند
ره زین شب تاریک نبردند برون
گفتند فسانه‌ای و در خواب شدند

Dedication

To my parents, Sedigheh and Naser, whose sacrifices made me everything I am.

To Sooshiant, whose unwavering support was always at my back.

To Jennifer, whose love sustained me through the challenging journey of Mathematics.

« آنان که محیط فضل و آداب شدند   در جمع کمال شمع اصحاب شدند

ره زین شب تاریک نبردند برون   گفتند فسانه‌ای و در خواب شدند »

عمر خیام نیشابوری

"The Revelations of Devout and Learn'd
Who rose before us, and as Prophets burn'd
Are all but Stories, which awoke from Sleep
They told their comrades, and to Sleep return'd"
Omar Khayyam.

Painting by Hossein Behzad (1965).

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I would like to thank my parents Sedigheh and Naser and my brother Sooshiant for their never ending sacrifice, their unconditional love and unshakable support. I owe all that I am to their existence.

Finally I would like to thank my wife Jennifer whose love and support sustained me through the challenging journey of life and mathematics.

Abstract

We study the magnetic Ginzburg–Landau energy in the critical coupling, also known as the abelian–Higgs model. It is known that entire solutions of the abelian–Higgs model blow down to (generalized) minimal submanifolds. We show that, in the so-called *multiplicity one regime*, critical points inherit an *improvement of flatness* and a *rigidity* property from their blow-down limit.

This thesis consists of three parts. In the first two parts we develop the necessary toolbox for part three.

First (in [44]), we develop a new class of weighted inequalities on any two-manifold (with boundary). Second (in [45]), using these inequalities and a selection principle (inspired by the quantitative isoperimetric inequality) we prove a sharp quantitative stability for the abelian–Higgs model in two dimensions.

Third, (in a collaboration with Guido De Philippis and Alessandro Pigati in [28]) we leverage these tool to develop a large scale regularity theory for the zero set of solutions (in the spirit of Allard's). In fact, in the multiplicity one regime we show the uniqueness of blowdowns. Then we classify solutions in dimensions $n < 5$ and local minimizers in all dimensions $n > 2$.

Contents

List of Figures

List of Appendices

Part I

Introduction & results

Chapter 1

Background

The area functional is perhaps one of the oldest functionals studied in mathematics. A fundamental problem in this context is Plateau's problem, which concerns the study of k-dimensional (generalized) manifolds with locally minimal k-dimensional area. Despite its seemingly simple formulation, the nonlinear nature of the k-area functional makes this problem highly challenging.

Like many other variational problem, this problem is split into two parts: *Existence* and *Regularity*. First we generalize and enlarge the space of k dimensional smooth manifolds enough so that we can find a solution through direct methods in calculus of variation. Then one starts working with a *weak* solution to deduce regularity properties and verify that the solution lives in a subset with richer/more regular structure than the abstract generalization.

Rather than attempting a comprehensive survey of the vast body of work on minimal surfaces, this chapter focuses on a specific methodology: obtaining minimal surfaces via *diffuse* approximations.

These models typically arise from physical phenomena describing phase transitions with a small diffusivity parameter $\varepsilon \to 0$. In such models, we study the transition layer of a critical point of a variational energy functional. This transition layer generally has a fixed co-dimension (determined by the dimension of the state space), and the energy is constructed to favor an ordered transition. In many cases, the bulk contribution of the energy is proportional to the k-dimensional area of the transition layer, which ultimately provides a diffuse approximation of minimal surfaces.

1.1 The Allen–Cahn model

For any $u : \Omega \to [-1, 1]$ for a domain $\Omega \in \mathbb{R}^n$ and a parameter $\varepsilon > 0$, the Allen–Cahn energy functional takes the following form:

$$E_\varepsilon(u) = \int_\Omega \varepsilon |du|^2 + \frac{(1-u^2)^2}{4\varepsilon} \,. \tag{1.1}$$

Note that while $\varepsilon|du|^2$ favors ordered transitions, $\frac{(1-u^2)^2}{4\varepsilon}$ favors the pure states ± 1.

Building on the pioneering ideas of De Giorgi, Modica [59], Ilmanen [49], and Hutchinson–Tonegawa [48], it has been understood that smooth critical points $u : M \to \mathbb{R}$ for the *Allen–Cahn energy* (here M could be a smooth manifold) serve as effective diffuse approximations of minimal hypersurfaces. The Allen–Cahn functional is a well studied model for phase transitions; a typical critical point u takes values in $[-1, 1]$, with $u \approx \pm 1$ (the pure phases) except in a transition region of thickness $\approx \varepsilon$, where most of the energy concentrates. Roughly speaking, this region is an ε-neighborhood of a minimal hypersurface, which acts as an interface between the two phases, and the energy density decays exponentially fast away from this interface. (See an illustration in Figure 1.1)

In particular, an application of the AM-GM inequality and co-area formula also shows that:

$$E_\varepsilon(u) = \int_\Omega \varepsilon |du|^2 + \frac{(1-u^2)^2}{4\varepsilon} \geq_{\text{AM-GM}} \int_\Omega |du|(1-u^2) = \int_{-1}^{+1} (1-t^2)\mathcal{H}^{n-1}\left(\{u = t\}\right) dt \,.$$

This understanding brought a novel, PDE-based way to attack variational problems for the co-dimension-one area [43], which often allows to obtain more refined results compared to other

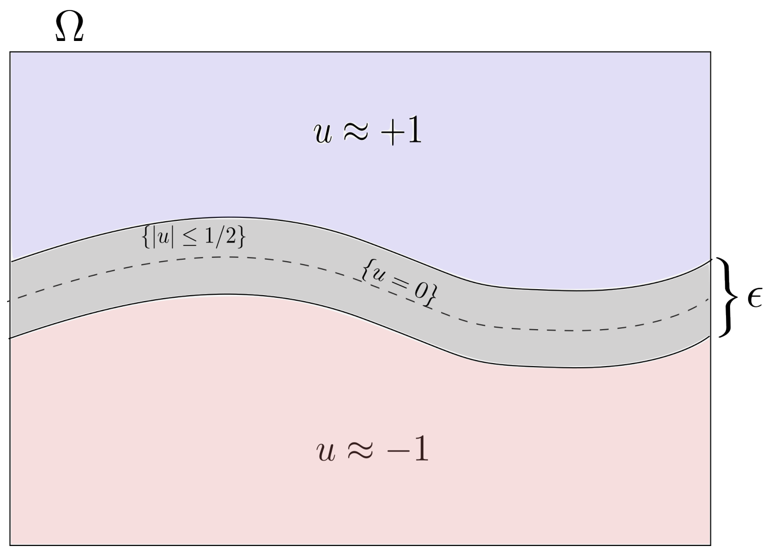

Figure 1.1: The transition layer/energy concentration set in the Allen–Cahn model

methods [17].

1.2 The non-magnetic Ginzburg–Landau model

In co-dimension two, similar attempts have been made by looking at the same energy for maps $u : M \to \mathbb{C}$, replacing u with $|u|$ in the second term:

$$E_\varepsilon(u) = \int_M \frac{|du|^2}{2} + \frac{(1-|u|^2)^2}{4\varepsilon^2} \,. \tag{1.1}$$

This corresponds to a simplified version of the Ginzburg–Landau model of superconductivity, popularized by Bethuel–Brezis–Hélein [9], where one neglects the magnetic field. The asymptotic analysis of this energy is substantially more involved, due to the lack of the aforementioned exponential decay. In fact [9] showed that for the case $M = B_1^2$ for small enough $\varepsilon > 0$ the set $\{|u| \leq \frac{1}{2}\}$ can be covered by a collection of balls as in Figure 1.2 and that the energy roughly expands as:

$$E_\varepsilon(u) \sim \pi|\log(\varepsilon)|\sum_{j=1}^{k} d_j^2 + \sum_{j<\ell} d_j d_\ell \left|\log|a_j - a_\ell|\right| + O(\text{Boundary confinement terms})\,. \tag{1.2}$$

Here $d_j = \deg(u, \omega_j)$ and:

$$\sum_{j=1}^{k} d_j = \deg(u, \partial B_1^2)\,.$$

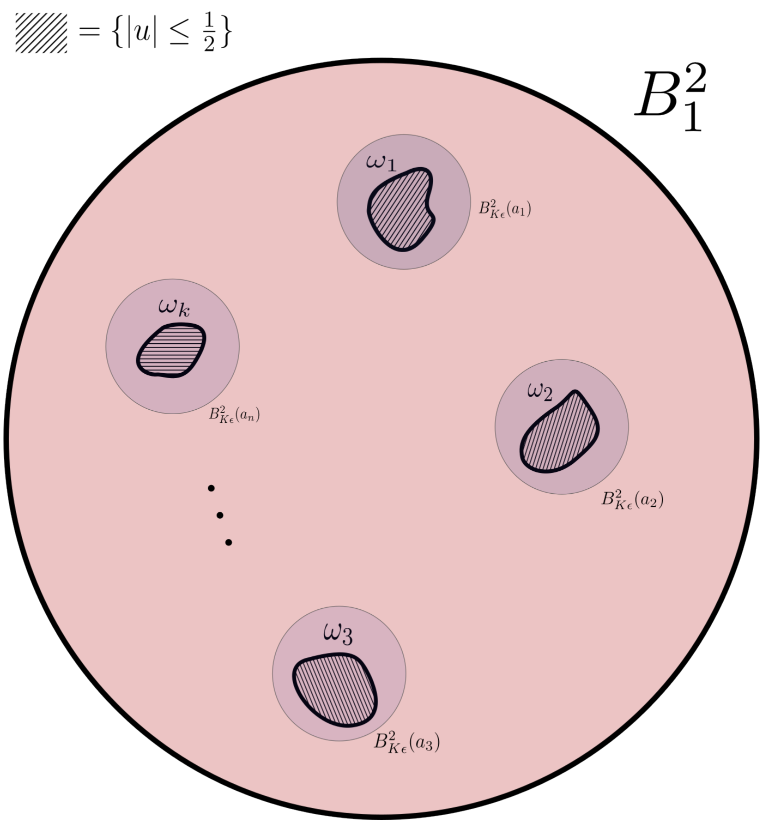

Figure 1.2: The vortex set in the Ginzburg-Landau model in two dimensions

To understand the $|\log(\varepsilon)|$-term, notice that a radial solution $u_0(x) = g(|x|)$ with one vanishing point in the origin roughly behaves like $\frac{z}{|z|}$ outside $B_{K\varepsilon}(0)$. This means that the energy grows as follows:

$$E_\varepsilon(u_0) \sim \int_{B_1 \setminus B_{K\varepsilon}} \left| d\left(\frac{z}{|z|}\right) \right|^2 = \pi |\log(\varepsilon)| \,.$$

One can phrase things in a different manner: Imagine $u = \frac{z}{|z|}$ is given on the boundary ∂B_1^2. Due to a simple topological obstruction, there does not exist a continuous extension of u in the interior with $|u| = 1$. This is also true if we require that $u \in W^{1,2}(B_1^2)$. The reason is the embedding $W^{1,2}(B_1^2) \hookrightarrow \mathrm{VMO}(B_1^2)$, where VMO stand for the space of functions with vanishing mean oscillation. In fact the authors in [14] show that this space support degree theory. One can see the Ginzburg–Landau energy (1.1) as the *ε-relaxation* of the extension problem.

In higher dimensions after re-normalizing the energy $\frac{E_\varepsilon(u)}{|\log(\varepsilon)|}$, the energy slowly concentrates on the *vortex set* as in Figure 1.3.

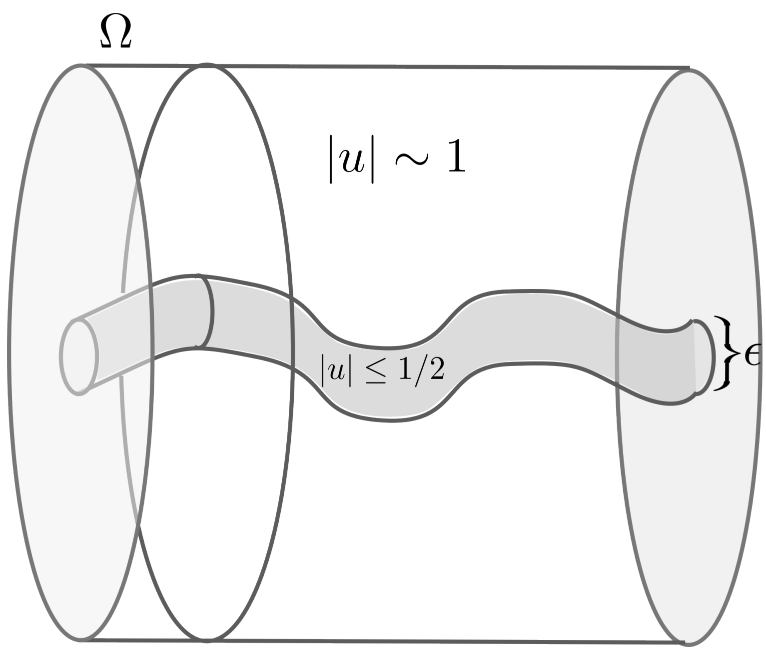

Figure 1.3: The vortex set in the Ginzburg–Landau (abelian–Higgs) model in dimensions $n > 2$

This slow decay brought mixed results: for instance, [8, 57] (using what they call *an η-compactness/ellipticity lemma*) showed that in the vanishing $\varepsilon \longrightarrow 0$ limit the energy concentrates on a stationary generalized manifold (varifold) and [22, 63] showed that the multiplicity of this varifold takes values in $\{1\} \cup [2, \infty)$ which could be non-integer. This fractional multiplicity is essentially due to the second term in the expansion (1.2) leaking into the first term when two components interact.

1.3 The abelian-Higgs model

On the other hand, including the magnetic field and looking at the so-called *self-dual regime* (also called *critical coupling*), we can consider the alternative energy

$$E_\varepsilon(u, \alpha) := \int_M \left[|du - i\alpha u|^2 + \frac{(1 - |u|^2)^2}{4\varepsilon^2} + \varepsilon^2 |d\alpha|^2 \right].$$

Apart from the different normalization, it differs from the previous energies by an additional variable, the one-form $\alpha \in \Omega^1(M; \mathbb{R})$, which twists the Dirichlet term and appears in the Yang–Mills term $|d\alpha|^2$ (indeed, the latter equals $|F_\nabla|^2$, where F_∇ is the curvature of the connection $\nabla := d - i\alpha$ on the trivial complex line bundle $\mathbb{C} \times M$).

Another way to see the purpose of this one-form (sometimes called the *magnetic potential*) is that for $u = re^{i\theta}$ and $\nabla = d - i\alpha$ for S^1-valued θ and real valued r and a one-form α we can write:

$$|\nabla u|^2 = |dr|^2 + r^2 |d\theta - \alpha| \, .$$

hence the *magnetic potential* or the one-form α helps the decay of energy by aligning itself with $d\theta$. Moreover the curvature term $|d\alpha|^2$ penalizes how much α can *twist.*

Note that for any $\xi \in C_c^\infty(M)$ the following transformation:

$$(u, \alpha) \rightsquigarrow (ue^{i\xi}, \alpha + d\xi) \, ,$$

leaves the energy unchanged. This group of symmetries are called *gauge transformations* and this property is referred to as *gauge invariance*. In particular, at a variational level, two *gauge equivalent* pairs are indistinguishable from one another.

This energy, in this specific self-dual regime (i.e., the choice of constants in front of each term), is well known in gauge theory, where it is often called $U(1)$*-Yang–Mills–Higgs*, or simply *abelian Higgs model*. It received a thorough treatment in dimension 2, with a complete classification of critical planar pairs (u, ∇) of finite energy by Taubes [74, 75] (discussed in detail in Part III). See also [47] for the case of Riemann surface and [12] for Kähler manifolds. Following an idea of Bogomolny [10], one can indeed see that on $\mathbb{R}^2$ for a section $u = re^{i\theta}$ and a connection $\nabla : d - i\alpha$:

$$\int_{\mathbb{R}^2} |du - iu\alpha|^2 + |d\alpha|^2 + \frac{(1-|u|^2)^2}{4} = 2\pi|N| + \int_{\mathbb{R}^2} |\star dr \pm r(\alpha - d\theta)|^2 + \left|\star d\alpha \mp \frac{1-r^2}{2}\right|^2 .$$

One immediately notices that minimizers must have the second integral pointwise zero, which are called the *vortex equations*:

$$\star dr = \pm r(\alpha - d\theta) \text{ and } \star d\alpha = \mp\frac{1-r^2}{2} .$$

Taubes showed that:

1. after prescribing the set $u^{-1}(0)$ counted with multiplicity, all solutions of the vortex equations are unique (up to a change of gauge).

2. All critical points on $\mathbb{R}^2$ are minimizers, meaning that they must satisfy the vortex equations.

With this information we can see that:

The abelian-Higgs model in $\mathbb{R}^2$ *models a system of* N *non-interacting particles.*

Recently, in [62], Stern and Pigati developed the asymptotic analysis in arbitrary Riemannian manifolds, obtaining the precise co-dimension-two analogue of the result by Hutchinson–

Tonegawa: see Theorem 13.1.1 below. Related facts, including Γ-convergence and the gradient flow convergence to mean curvature flow, have also been verified, by Parise, Stern, and Pigati [60, 61]. In fact one can summarize the results of [60–62] as:

The abelian-Higgs model is an effective diffuse approximation for the $n-2$-area functional.

1.4 De-Giorgi's conjecture on the Allen–Cahn model

Since the work of De Giorgi [24] and Allard [2], it is known that almost-flat minimal submanifolds enjoy an *improvement of flatness*, i.e., they become even closer to a plane at smaller scales, in a quantitative way. Iteration of this improvement of flatness is the key mechanism in proving (quantitative) regularity of minimal surfaces. The key analytical fact behind this decay property is the observation that the linearization of the minimal graph equation is the Laplace equation, whose solutions enjoy similar decay properties.

A related question, in the spirit of the classical Liouville theorem, is whether globally defined objects should be planar. The famous *Bernstein's conjecture* predicts that this is always true for minimal graphs $\mathbb{R}^{n-1} \to \mathbb{R}$, which are automatically (locally) area-minimizing hypersurfaces. In view of the improvement of flatness, this question quickly reduces to understanding whether any blow-down is necessarily a hyperplane. Bernstein's question was answered affirmatively by the works of Fleming, De Giorgi, Almgren, and Simons for $n \leqslant 8$, while Bombieri–De Giorgi–Giusti produced a counterexample for $n = 9$, whose blow-down corresponds to the Simons cone, in [11].

It is well-known by now that in the Allen–Cahn model, the set $\{u = 0\}$ behaves like a minimal surface on large scales. By analogy, De Giorgi conjectured that critical points $u : \mathbb{R}^n \to \mathbb{R}$ of the Allen–Cahn energy further inherit a *graphical Bernstein-type* property from minimal surfaces:

Conjecture 1.4.1 (De-Giorgi)**.** *Let $u : \mathbb{R}^n \to [-1, 1]$ be a bounded critical point of the Allen–Cahn energy (1.1), which is monotone in one direction, say $\partial_n u > 0$. Then at least when $n \leq 8$, the solution*

u must be one-dimensional, as in:

$$u(x) = \tanh(x.a - b)\,,$$

for $b \in \mathbb{R}$ and $a \in \mathbb{R}^n$ with $|a| = 1$ and $a_N > 0$.

The question has been solved by Ghoussoub–Gui for $n = 2$, in [42], by Ambrosio–Cabré for $n = 3$, in [4], and by Barlow–Bass–Gui under additional regularity for the level sets, in [6]. Finally, in [70] Savin settled the conjecture for all $n \leqslant 8$ under the assumption that $u(x', x_n) \to \pm 1$ as $x_n \to \pm\infty$, for any fixed $x' \in \mathbb{R}^{n-1}$. In fact he proved the following theorem:

Theorem (Savin [70]). *Let $u : \mathbb{R}^n \to [-1, 1]$ be a bounded critical point of the Allen–Cahn energy (1.1), which is monotone in one direction, say $\partial_n u > 0$ and:*

$$\forall x' \in \mathbb{R}^{n-1} : \quad \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1\,.$$

Then the solution u must be one-dimensional, as in:

$$u(x) = \tanh(x.a - b)\,,$$

for $b \in \mathbb{R}$ and $a \in \mathbb{R}^n$ with $|a| = 1$ and $a_N > 0$, provided that $n \leq 8$.

The general outline of the arguments are the following two steps:

1. First with a simple compactness argument we can see that the configuration is close to be flat on large scales, with respect to a possibly changing plane.

2. The second step is an improvement of flatness: if the configuration is close enough to be flat on scale 1, Then it is much closer to be flat at scale $1/2$.

Iterating step 2 on large scales we can deduce rigidity/regularity. The precise definition of *flatness* depends on the problem however the intuition comes from an idea of De-Giorgi that the area functional linearizes to the Laplace's equation. We can see this in the space case of graph $f : \Omega \to \mathbb{R}$ with small $\mathrm{Lip}(f) \ll 1$, using the Taylor expansion $\sqrt{1+x^2} = 1 + \frac{x^2}{2} + O(x^3)$ to see that the area functional is a third-order approximation of the Dirichlet energy:

$$\mathrm{Area}(\mathrm{graph}(f)) = \int_\Omega \sqrt{1+|df|^2} \sim \int_\Omega 1 + \frac{|df|^2}{2} = |\Omega| + \mathrm{Dir}(f)\,,$$

whose minimizers enjoy good decay properties.

As for minimal graphs, De Giorgi's conjecture (even with the extra assumption used by Savin) is false for $n \geqslant 9$: a counterexample has been constructed by Del Pino–Kowalczyk–Wei, in [29].

Savin's approach uses viscosity techniques, resembling the Krylov–Safanov theory in spirit. In particular, while his groundbreaking methods have a wide range of applicability, even beyond variational equations, it is not always clear how one can extend these techniques to the vectorial setting (higher do-codimension), where the maximum principle does not apply; see however [27, 69].

Recently, Wang [80] obtained a variational proof of Savin's theorem, following the strategy of Allard's proof of excess decay for stationary varifolds:

Theorem (Wang [80])**.** *Let $u : \mathbb{R}^n \to [-1,1]$ be a bounded critical point of the Allen–Cahn energy (1.1), with the energy bound:*

$$\lim_{R\to\infty} \frac{1}{R^{n-1}} \int_{B_R} |du|^2 + \frac{(1-u^2)^2}{4} \leq \sigma_0 + \tau\,,$$

where $\sigma_0 = \int_{-1}^{+1}(1-t^2)$. Then the solution u must be one-dimensional, as in:

$$u(x) = \tanh(x.a - b)\,,$$

for $b \in \mathbb{R}$ and $a \in \mathbb{R}^n$ with $|a| = 1$ and $a_N > 0$, provided that $n \leq 8$ and τ is chosen small enough.

Wang's paper has been the starting point for our investigation of the regularity properties of the zero set of solutions of the Yang–Mills–Higgs equations.

Chapter 2

Contributions

Building on [60–62] and inspiring from [70, 80] we ask the following question:

Question: *Do critical points (minimizers) of the abelian-Higgs model inherit any rigidity/regularity from minimal surfaces?*

The main contribution of this thesis can be summarized vaguely in the following theorem:

Answer: ([28, 44, 45]) *In the multiplicity one regime, stationary pairs and minimizers of the abelian-Higgs energy enjoy an improvement of flatness.*

However to prove the above theorem, one needs to build up a tool-box. In Part II and Part III of the thesis we build this machinery and using these tools we attack the main problem in Part IV.

2.1 New weighted inequalities on two-manifolds

In Part II we deal with a family of weighted elliptic inequalities on two-manifolds obtained in [44]. These estimates are a crucial tool in our analysis later on and will be invoked many times in Parts IIIIV; Moreover they can be seen as generalizations to the Cafarelli-Kohn-Nirenberg interpolations estimates [15] or a class of Carleman-type estimates in dimension 2. The main motivation is a *weighted Hodge decomposition.*

2.1.1 Results

We provide L^2-weighted elliptic estimates for a class of positive weights $\omega \in W^{1,2}(\mathcal{M}^2)$ on smooth Riemannian connected two-manifolds $(\mathcal{M}^2, g)$ that weakly satisfy

$$\omega^2 \Delta_g \log(\omega) = -\kappa(x)\omega^2,$$

Examples of weights: In the case of $M := \mathbb{R}^2$ all weights of the form

$$\omega(x) = \Pi_{k=1}^{n} |x - a_k|^{\alpha_k} \text{ with } \{a_k\}_{k=1}^{n} \subset \mathbb{R}^2 \text{ and } \alpha_k > 0\,,$$

are admissible. More generally (as in [44]) for a smooth open and bounded domain $\Omega \subset M$ in a smooth two-manifold, the weights can take the following form:

$$\omega(x) = \Pi_{k=1}^{n} e^{-\alpha_k G_{p_k}(x)} \text{ with } \{p_k\}_{k=1}^{n} \subset \Omega \text{ and } \alpha_k > 0\,,$$

where $G_p(x) = G(p, x)$ is the Green's function for the domain Ω centered on p, namely the fundamental solution for the Laplacian on Ω (for a comprehensive account of the Green's function on smooth manifolds see [56]). Following the observation in [56, eq (1.1)], we see that there is some constant $C > 0$ such that any weight of the form (3.5) satisfies:

$$C^{-n}\omega(x) \leq \Pi_{k=1}^{n} d(x, p_k)^{\alpha_k} \leq C^n \omega(x) \text{ in } \Omega\,,$$

where $d(x, y)$ is the geodesic distance between x, y on M.

The most important contribution of [44] is the following theorem:

Theorem 2.1.1 (Theorem 3.1.3)**.** *Let $(\mathcal{M}^2, g)$ be a smooth 2-manifold and a weight ω in Definition 4.0.1 with $\kappa = 0$ and $\varepsilon \geq 0$ and $\Omega \subset \mathcal{M}^2$ a smooth open domain. Then for any function*

$f \in C_c^\infty(\Omega)$ we have that:

$$\int_\Omega \omega^{2+2\varepsilon} |\nabla f|^2 d\mathrm{vol}_g \le C \frac{(\sup_\Omega \omega)^{2\varepsilon}}{\varepsilon^2} \int_\Omega \frac{\omega^4}{|\nabla \omega|^2} |\Delta_g f|^2 d\mathrm{vol}_g \,,$$

with the bound $C \le \frac{8\varepsilon^2 + 5(1+\varepsilon)^4}{8(1+\varepsilon)^2}$ which is comparable to $\frac{5}{8}$ as $\varepsilon \to 0$.

The main motivation of the above theorem is in fact the following estimate regarding the *weighted Hodge decomposition*:

Corollary 2.1.2 (Lemma 3.0.1)**.** *Let $(\mathcal{M}^2, g)$ be a Riemannian 2-manifold and let $\Omega \in \mathcal{M}^2$ be a smooth open domain and ω is a weight as in Definition 4.0.1 with $\kappa = 0$. Any smooth one-form $A \in C_c^\infty(\bigwedge^1 \Omega)$ has a Hodge decomposition and a weighted Hodge decomposition as follows:*

$$A = \star d\xi_1 + d\xi_2 \text{ and } \omega A = \star \omega d\phi_1 + \omega^{-1} d\phi_2 \,,$$

for 4 compactly supported functions $\xi_1, \xi_2, \phi_1, \phi_2$. Moreover for any $0 \le \varepsilon \le C$ we have the estimates:

$$\|\omega^{1+\varepsilon} d(\xi_1 - \phi_1)\|^2_{L^2(\Omega)} \le C \frac{(\sup_\Omega \omega)^{2\varepsilon}}{\varepsilon^2} \|\omega^{-1} d\phi_2\|^2_{L^2(\Omega)} \,.$$

2.1.2 Overview

The main motivation of these estimates is in fact the *weighted Hodge decomposition* of Lemma 3.0.1. To understand better, we need to adapt the Hodge decomposition to a weighted case.

What is Hodge decomposition of a one-form A? It is the variational problem of finding the nearest closed (co-closed) form to A. To adapt this to the weighted case we look at the following energy:

$$\int_{B_1^2} |x|^2 |A - d\phi|^2 \,, \tag{2.1}$$

In an appropriate family of function we can find a minimizer of (2.1) by the direct method in the calculus of variation. Naturally we define:

$$X = \{\phi \in C_c^\infty(B_1^2) : \int_{B_1^2} |x|^2 |d\phi|^2 < \infty\},$$

and an inner product:

$$\langle \phi_1, \phi_2 \rangle_X := \int_{B_1^2} |x|^2 \langle d\phi_1, d\phi_2 \rangle .$$

Then we look at the completion $\overline{X}$ with respect to the norm $\|.\|_X$ induced by the inner product. A special case of a well-known class of inequalities called Caffarelli-Kohn-Nirenberg interpolation inequalities [15] is as follows:

$$\forall f \in C_c^\infty(\mathbb{R}^2) : \quad \int_{\mathbb{R}^2} |f|^2 \leq \int_{\mathbb{R}^2} |x|^2 |df|^2 .$$

This asserts that the space X is equivalent to the set $\{u$ such that $|x|u \in W^{1,2}\}$.

In the paper [16] the authors recast these inequalities in a different light. In fact it turns out that after a log-polar transformation $B_1^2 \ni x \rightsquigarrow (-\log(|x|), \theta) \in [0, \infty) \times S^1$ with $u = |x| f$ the weighted term translates to:

$$\int_{\mathbb{R}^2} |x|^2 |df|^2 = \int_{S^1 \times [0,\infty)} |u|^2 + |du|^2 \, d\mathrm{vol}_{S^1 \times [0,\infty)} .$$

Using weak lower semi-continuity in Sobolev spaces (either in the cylinder or on the plane itself using CKN inequalities), we can apply the direct method in the calculus of variations to assert that (2.1) indeed has a minimizer ϕ. The Euler–Lagrange equation becomes:

$$d^*(|x|^2(A - d\phi)) = 0 .$$

This means that $A - d\phi$ is co-closed, since it has zero trace we can see that:

$$|x|A = |x|d\phi + |x|^{-1} \star d\xi\,.$$

for some compactly supported function ξ. In fact the following orthogonality relation holds:

$$\int_{B_1^2} |x|^2|A|^2 = \int_{B_1^2} |x|^2|d\phi|^2 + |x|^{-2}|d\xi|^2\,.$$

Now consider the Hodge decomposition of $A = dq + \star dp$ and the weighted one $|x|A_j = |x|d\phi + |x|^{-1} \star d\xi$. The main result of the paper can be vaguely stated as the following estimate:

$$\int_{B_1^2} |x|^{2+2\varepsilon}|d(\phi - q)|^2 \leq C\varepsilon^{-2} \int_{B_1^2} |x|^{-2}|d\xi|^2\,.$$

In fact in [44] we show that the closed part of the weighted and the non-weighted decomposition are close in a precise quantitative way.

In [44] we only need the following assumption on the weights:

$$\omega^2 \Delta \log(\omega) = 0\,.$$

This permits us to generalize to weights vanishing at multiple points. Moreover the proofs are (careful however) simple integration by parts, resulting in uniform constants.

2.2 Quantitative stability of Yang–Mills–Higgs instantons in dimension 2

Since the work of Taubes [74, 75] and Bogomolny [10] it is known that an N-vortex pair (u, α) on $\mathbb{R}^2$ enjoys the lower bound:

$$E(u, \alpha) \geq 2\pi|N|$$

for the Yang–Mills–Higgs energy (5.1) with $\varepsilon = 1$ (here we identify $\nabla : d - i\alpha$). Moreover Taubes showed that in the case of equality $E(u, \alpha) = 2\pi|N|$, after prescribing the zero set $u^{-1}\{0\}$ counted with multiplicity, the solution is unique (up to change of gauge).

To study the fine properties of the zero set of critical points in higher dimensions, it is necessary to have a fine understanding of how *nearly minimal* solutions behave. In fact along the concentration set (the limiting generalized $n-2$ submanifold), perpendicular 2-slices are close to minimizers of the energy. The estimates derived in this paper heavily rely on [44] and are instrumental in [28].

2.2.1 Results

The most important contribution of [45] is to improve the uniqueness of Taubes [74] to a sharp quantitative stability in the following theorem:

Theorem (Theorem 5.1.1). *For any integer N there exists a constant $C_{|N|} > 0$ such that for any section and connection $(u, \nabla) \in W^{1,2}_{loc}(\mathbb{R}^2)$ on the trivial line bundle $L = \mathbb{R}^2 \times \mathbb{C} \to \mathbb{R}^2$ with $\deg(u) = N$ and small enough discrepancy $E(u, \nabla) - 2\pi|N|$ we have that:*

$$\min_{(u_0, \nabla_0) \in \mathcal{F}} \|u - u_0\|^2_{L^2(\mathbb{R}^2)} + \|F_\nabla - F_{\nabla_0}\|^2_{L^2(\mathbb{R}^2)} \leq C_{|N|} \left[E(u, \nabla) - 2\pi|N|\right],$$

where (up to a conjugation) $\mathcal{F}$ is the family of all N-vortex minimizers of the Yang-Mills-Higgs energy.

Some care has been done in the proof to make the constant on the right hand side depend only on $|N|$ and reduce the assumptions, which we will elaborate in the following section.

2.2.2 Overview

First we start with the identity:

$$E(u,\nabla) - 2\pi|N| = \int_{\mathbb{R}^2} r^2|\star d\log(r) + (\alpha - d\theta)|^2 + |\star d\alpha - \frac{1-r^2}{2}|^2,$$

where $u = re^{i\theta}$ for positive real valued r and S^1-valued θ. Assume the pair (u,α) is smooth enough and take a solution of the Taubes [74] vanishing on $u^{-1}\{0\}$ (which is unique up to a change of gauge), namely (u_0,α_0). Now after a change of gauge we can assume that u and u_0 have the same phase. Now defining:

$$h = \log(\frac{|u|}{|u_0|}) \text{ and } \beta = \alpha - \alpha_0\,,$$

we see that (after a linearization):

$$E(u,\nabla) - 2\pi|N| \sim \int_{\mathbb{R}^2} \omega^2|\star dh + \beta|^2 + |\star d\beta + |u_0|^2 h|^2\,. \tag{2.1}$$

Note that in the absence of the weight r^2 we can carry out a Hodge decomposition and a standard compactness argument to bound $\int_{\mathbb{R}^2} |h|^2$ with the expression above. In the weighted case we leverage inequalities of [44]. Precisely we perform a weighted and a standard Hodge decomposition on β:

$$\beta = \star d\xi_1 + d\xi_2 \text{ and } \omega\beta = \omega \star d\phi_1 + \omega^{-1} d\phi_2\,.$$

Plugging both into (2.1) we see that:

$$E(u,\nabla) - 2\pi|N| \sim \int_{\mathbb{R}^2} \omega^2 |d(h+\xi_1)|^2 + \omega^{-2}|d\xi_2|^2 + |\Delta\phi_1 + |u_0|^2 h|^2\,.$$

Using a compactness argument we aim to bound $\int_{\mathbb{R}^2} \omega^2|h|^2$ however to close up the argument we need to estimate the distance of ξ_1 and ϕ_1. This is the main use of the estimates developed in [44], in particular Lemma 3.0.1 which asserts that:

$$\int_{\mathbb{R}^2} \omega^{2+2\varepsilon}|d\xi_1 - d\phi_1|^2 \leq \frac{C}{\varepsilon^2}\int_{\mathbb{R}^2} \omega^{-2}|d\xi_2|^2\,.$$

With some additional details we conclude stability for regular enough pairs. However for a generic pair one does not expect that r is comparable to any admissible weight (3.4). To generalize the stability to all pairs with small enough discrepancy, we perform a selection principle inspired by [19] using a penalized functional. In fact this is equivalent to running the gradient flow for one second.

2.3 Decay of excess for the abelian–Higgs model

We study critical pairs (u,∇) of the Yang–Mills–Higgs energy:

$$E_\varepsilon(u,\nabla)[\Omega] = \int_\Omega |\nabla u|^2 + \varepsilon^2|F_\nabla|^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2}\,.$$

Here $\Omega \subset \mathbb{R}^n$ is a subset of the flat space and $(u, \nabla : d - i\alpha)$ is a section and connection on the trivial line bundle $\mathbb{C}\times\mathbb{R}^n$. We recall the main result of [62]:

Theorem ([62] Theorem 1.1)**.** *Let $L \longrightarrow M$ be a Hermitian line bundle over a closed, oriented Riemannian manifold M^n of dimension $n \geqslant 2$, and let $(u_\varepsilon, \nabla_\varepsilon)$ be a family of critical pairs for the Yang–Mills–Higgs energy E_ε (5.1) satisfying a uniform energy bound:*

$$E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \leq \Lambda < \infty \,.$$

Then as $\varepsilon \longrightarrow 0$, the energy measures

$$\mu_\varepsilon := \frac{1}{2\pi} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \mathrm{vol}_g \,,$$

converge sub-sequentially, in duality with $C^0(M)$, to the weight measure μ of a stationary integral $(n-2)$-varifold V. Moreover the currents Γ_ε dual to the curvature $\frac{1}{2\pi}F_\varepsilon$ and Jacobian two-form $J(u_\varepsilon, \nabla_\varepsilon) = \frac{1}{2\pi} d\langle \nabla u, iu \rangle$ converge sub-sequentially to an integral $(n-2)$ current Γ with $|\Gamma| \leq \mu$.

The results above vaguely states that, the energy of the critical pair concentrates around (generalized) sub-manifolds of co-dimension 2. Allard regularity theorem [2] asserts that the limit varifold is indeed regular around points of multiplicity one.

In Part IV we investigate if critical pairs $(u_\varepsilon, \nabla_\varepsilon)$ in the *multiplicity one regime* inherit any rigidity/regularity from their limit.

2.3.1 Results

Blow-up analysis around a multiplicity one regime, leads us to investigate pairs (u, ∇) on the trivial line bundle $\mathbb{C} \times \mathbb{R}^n$ with the energy bound:

$$\lim_R \frac{1}{\omega_{n-2} R^{n-2}} \int_{B_R} \left[|\nabla u|^2 + |F_\nabla|^2 + \frac{(1-|u|^2)^2}{4} \right] \leq 2\pi + \tau \,, \tag{2.1}$$

for small enough $\tau > 0$.

We measure flatness with respect to an $n-2$ plane $S = \mathrm{span}\{e_3, \dots, e_n\}$ in two ways (see (14.1)):

$$\mathbf{E} := \mathbf{E}_1 + \mathbf{E}_2 \, .$$

The first $\mathbf{E}_1$ is an unoriented excess and parallels Allard's L^2 tilt-excess in the setting of integral varifolds:

$$\mathbf{E}_1(u, \nabla, B_r(x), S) := \frac{r^{2-n}}{2\pi} \int_{B_r(x)} \left[\sum_{k=3}^{n} |\nabla_{e_k} u|^2 + \varepsilon^2 \sum_{(j,k)\neq(1,2)} \omega(e_j, e_k)^2 \right] .$$

The second excess $\mathbf{E}_2$ depends on orientation:

$$\mathbf{E}_2(u, \nabla, B_r(x), S) := \frac{r^{2-n}}{2\pi} \int_{B_r(x)} \left[|\nabla_{e_1} u + i \nabla_{e_2} u|^2 + \left| \varepsilon\omega(e_1, e_2) - \frac{1-|u|^2}{2\varepsilon} \right|^2 \right],$$

and it measures how far slices are, from solutions of the vortex equations (6.4).

The full excess $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ parallels that of De-Giorgi for integral currents. Around points of multiplicity one (2.1), we apply a blow-down and use a compactness argument combined with Allard's regularity theorem [2] to conclude that: $\mathbf{E}$ vanishes on large scales $R \to \infty$, with respect to a possibly changing plane. The main results of Part IV is to upgrade this to an *improvement of flatness* type result, or precisely a *decay of excess* while controlling the tilt.

Vaguely for *general critical pairs* we prove:

Theorem (Theorem 14.2.2)**.** *Critical pairs in the multiplicity one-regime (2.1) with small enough $\tau > 0$ enjoy a decay of the first excess $\boldsymbol{E}_1$, until the scale R, where $\boldsymbol{E}_1 \lesssim R^{-2} |\log \boldsymbol{E}|^2 \sqrt{\boldsymbol{E}}$.*

An immediate corollary of Theorem 14.2.2 is:

Corollary (Corollary 10.3.4)**.** *The blow-down of critical pairs as above is unique.*

More precisely:

Corollary (Theorem 10.3.5). *The zero set of a critical pair as above (with the bound (2.1)) is in a $C\varepsilon^{\frac{1}{1+\alpha}}$ neighborhood of a graph $f : \mathbb{R}^{n-2} \to \mathbb{R}^2$ with $\|f\|_{C^{1,\alpha}} \ll 1$ (after a possible rotation).*

Iterating Theorem 14.2.2, we also get the following rigidity corollary for low dimensions :

Corollary (Theorem 10.3.7). *For $2 \leq n \leq 4$ there exists $\tau > 0$ such that all entire critical pairs (u, ∇) with the multiplicity one energy bound (2.1) are two-dimensional; meaning that there exists a projection $P : \mathbb{R}^n \to \mathbb{R}^2$ and a one-vortex solution of Taubes (u_0, ∇_0) such that $(u, \nabla) = P^*(u_0, \nabla_0)$.*

Theorem 10.3.7 reduces classification result to a Gibbon's conjecture in Conjecture 10.3.6 (currently open for critical pairs in dimension $n \geq 5$).

We can leverage *diffuse regularity* of Theorem 10.3.5 to prove more for *locally minimizing* pairs:

Theorem (Theorem 10.3.8). *For any number $\beta > 0$ and dimension n, there is a threshold $\tau(\beta, n) > 0$ such that locally minimizing pairs in the multiplicity one regime (2.1) with $\tau = \tau(\beta, n)$, enjoy a decay of the full excess $\mathbf{E}$, until the scale R where $\mathbf{E} \lesssim R^{-\beta}$*

Additionally, we leverage this stronger decay of excess to prove rigidity for local minimizers in all dimensions:

Corollary (Theorem 10.3.7). *For any dimension $n > 0$, there exists a $\tau(n) > 0$ such that all entire locally minimizing pairs in the multiplicity one regime (2.1) with $\tau = \tau(n)$, are two dimensional.*

2.3.2 Overview

It is conceptually easier to look at blow-downs; meaning studying the problem when $\varepsilon \to 0$.

The strategy for the proof of Theorem 10.3.2 can be summarized into three steps:

1. **Lipschitz approximation** in Proposition 15.2.1: We show that the location of the zero set $u^{-1}(0)$ (after a possible rotation), is well approximated by a Lipschitz graph $h : B_1^{n-2} \to B_1^2$,

with a good estimate on the Dirichlet energy:

$$\int_{B_1^{n-2}} |Dh|^2 \leq C\mathbf{E}_1$$

Here because of gauge invariance (unlike [80]) we do not have access to arbitrary level sets of u. Instead we slice at the current Γ_ε dual to the Jacobian $J(u, \nabla)$ and test each slice with a function $\psi \in C_c^1(B_1^2)$ to define (through a duality relation in Definition 15.1.1):

$$\langle \Phi_\psi(y), \phi(y) \rangle = \langle \Gamma, \psi(z)\phi(y)dy \rangle \,,$$

for $\mathbb{R}^n \ni x := (y, z) \in \mathbb{R}^{n-2} \times \mathbb{R}^2$. Then using modified Jerrard–Soner [54] type estimates and standard BV-theory we derive the desired Lipschitz approximation. Moreover we see that $\frac{h-\left(\int_{B_1} h\right)}{\sqrt{\mathbf{E}_1}}$ is *weakly* pre-compact in $W^{1,2}$.

2. **Harmonic approximation** in Proposition 16.1.2: We can use inner variations (domain variations) to show that a symmetric matrix valued function $T_\varepsilon(u, \nabla)$ (defined in (13.3)), called *the stress energy tensor* is row-wise divergence free. In the context of Generalized varifolds of Ambrosio–Soner, T_ε is a stationary varifold.

   Moreover the self-dual structure (quantified by $\mathbf{E}_2$) asserts that $T_\varepsilon(u, \nabla)$ is L^2-close to $J(u, \nabla)$ on certain non-diagonal elements.

   Vaguely this shows that the oriented structure of the current Γ and the unoriented structure of the stationary varifold T_ε are closely related.

   Using this observation and slicing formulas, we show that (testing T_ε with appropriate domain variations) that:

$$\left| \int_{B_1^{n-2}} Dh.D\xi \right| \leq C\sqrt{\mathbf{E}\mathbf{E}_1} \|D\xi\|_\infty \,,$$

for any test function $\xi \in C_c^\infty(B_1^{n-2})$. Naming $\tilde{h} = \frac{h-\left(\int_{B_1} h\right)}{\sqrt{\mathbf{E}_1}}$, we see that:

$$\int_{B_1^{n-2}} |D\tilde{h}|^2 \leq C \text{ and } \left|\int_{B_1^{n-2}} D\tilde{h}.d\xi\right| \leq C\sqrt{\mathbf{E}}\,.$$

This shows that any weak $W^{1,2}$ limit of $\tilde{h}$ is in fact harmonic; meaning there is some harmonic function w such that:

$$\int_{B_1^{n-2}} |h-w|^2 \leq o(\mathbf{E}_1)\,.$$

3. **Caccioppoli-type estimate and decay of $\mathbf{E}_1$** in Proposition 16.2.1: The beginning of the regularity theory of elliptic equations is the Caccioppoli inequality, also called the *excess-height* inequality in the context of minimal surfaces. Vaguely stated, we test the varifold associated to T_ε with appropriate domain variations to derive an excess height inequality:

$$\int_{B_1^{n-2}} \phi^2 \mathbf{E}_1 \leq C \int_{B_1^{n-2}} \Delta\phi^2 (|h-c|^2 + \text{Variance of slice measure})\, dy\,.$$

Then we use decay properties of the harmonic approximation for the height, and conclude the decay for the first term.

For the variance term, we use the quantitative stability of Theorem 5.1.1, we show that for most slices:

$$\left|\text{Variance of slice measure} - \varepsilon^2 v_0^2\right| \lesssim C|\log \mathbf{E}|^2\sqrt{\mathbf{E}}\,,$$

where v_0^2 is the variance of the measure associated to a one-vortex solution of Taubes. Putting everything together we obtain Theorem 10.3.2.

For local minimizers, we construct competitors to improve the decay. In fact we show that in this case, the full excess in strongly approximated by the Dirichlet energy:

$$\int_{B_1^{n-2}} \left|\mathbf{E}_y - |dw|\right|^2 \lesssim o(\mathbf{E})\,.$$

Vaguely speaking, we update the weak $W^{1,2}$ pre-compactness to a strong one. In other words, we show that there is no concentration of excess $\mathbf{E}$ on arbitrary small (bad) sets. We assume contradiction:

- Either some excess $\mathbf{E}$ concentrates on an arbitrary small set, or.
- The excess is not strongly $W^{1,2}$ approximated by a harmonic function.

Both of the conditions above mean that one can replace h in the interior with a function that has less excess $\mathbf{E}$. Minimality should tell us that this is impossible. Then we use this information to construct a competitor to reach contradiction. This construction is the most technical part of the thesis and we summarize it in two main steps:

1. **The Pull-back pair** in Proposition 19.1.1: First we pull-back a one-vortex solution along the graph of a competitor. The energy of this pull-back pair is effectively its Dirichlet energy, however it is not yet a competitor as boundary conditions are not met. (see Figure 2.1)

   Now we need to interpolate between the constructed pull-back solution and the boundary condition, while carefully controlling estimates. This is the main use of quantitative stability results in Part III (the sharp power is what unlocks all dimensions $n \geq 2$). However nearly minimizing slices are close to solutions in *some gauge*. In three dimensions (see Figure 2.1), we only need to perform a gauge transformation on two slices. However in higher dimensions, this becomes very complicated.

2. **A good gauge** in Proposition 19.2.1: Fixing a gauge is a *measurement*, and if it's done on smaller scales, it will yield better estimates. The intrinsic scaling of this problem is ε,

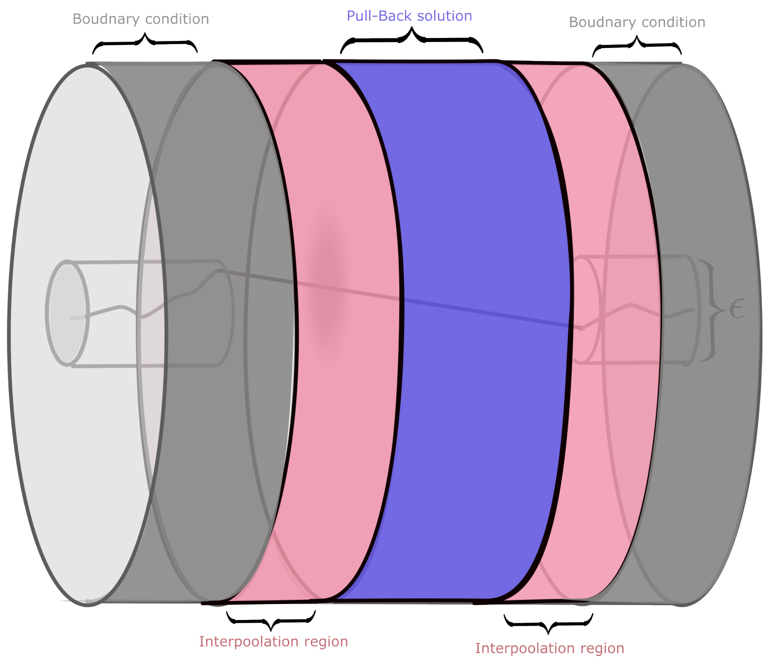

Figure 2.1: The pull-back pair

which means in order to expect any good estimates, we need to be gauge fixing at the ε-level. However we need to do this globally, which poses a problem. In order to overcome this difficulty we proceed in the following steps:

(a) We cover the vortex set $\{|u| \leq \frac{1}{2}\}$ with thin cylinders of size $\left(B^2_{C\beta|\varepsilon \log \varepsilon|} \times B^{n-2}_{C\varepsilon}\right)$ as in Figure 2.2. The *diffuse regularity* in Theorem 10.3.5 ensures that no two cylinders are on top of each other.

(b) In each cylinder, we find a careful gauge transformation ξ_k by solving (19.7). At this step, the shape of the cylinders costs us extra $|\log \varepsilon|$ factors in the estimates. However this is enough for our purpose.

(c) For overlapping cylinders $j < k$ we then estimate:

$$\|d\xi_k - d\xi_j\|^2_{L^2(\text{overlap region})} \leq \mathbf{E}(\text{overlap region})\,. \tag{2.2}$$

(d) Then we use a partition of unity ϕ_k for the thin cylinders to attach all ξ_ks together.

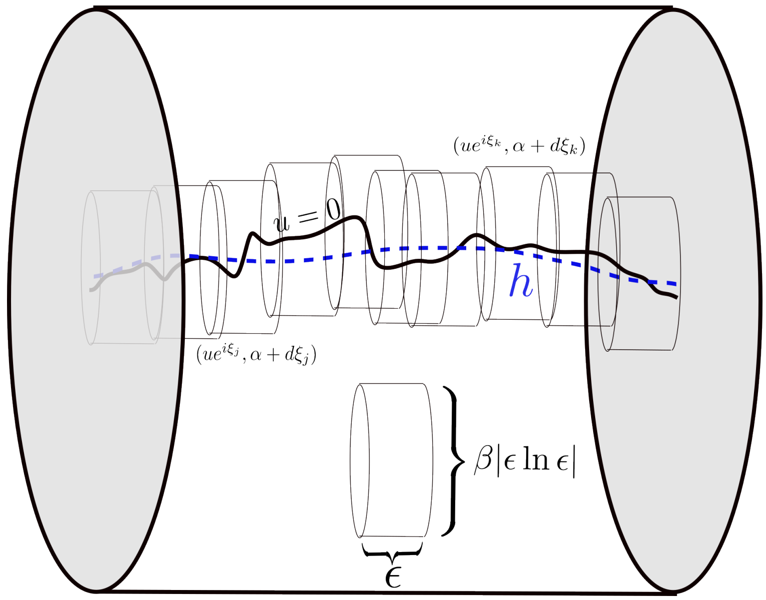

Figure 2.2: A good gauge: the covering in the interpolation region

Note that $|d\phi| \lesssim \varepsilon^{-1}$.

Now since $\sum \phi_j = 1$ and $\sum_j d\phi_j = 0$ we can rewrite:

$$d \sum_j \phi_j \xi_j = \sum_j d\phi_j \xi_j + \phi_j d\xi_j = \sum_{j,k} \phi_k d\phi_j (\xi_j - \xi_k) + \sum_j \phi_j d\xi_j \,.$$

Hence we can estimate:

$$\|d \sum_j \phi_j \xi_j\|_{L^2} \leq C \sum_j \|\xi_j\|_{L^2(j\text{th domain})} + C\varepsilon^{-1} \sum_{\text{overlapping } j<k} \|\xi_j - \xi_j\|_{\text{overlap of } j,k}$$

However note that the second term can be bounded using Poincaré inequality:

$$\|\xi_j - \xi_j\|_{\text{overlap of } j,k} \leq |\varepsilon \log \varepsilon| \|d\xi_j - d\xi_k\|_{L^(\text{overlap})} \leq \mathbf{E}(\text{overlap of } j, k) \,.$$

This is the key estimate that allows us to find a good gauge to interpolate and estimate.

(e) Away from the vortex set, the energy decays like:

$$e^{-\beta\varepsilon|\log\varepsilon|/\varepsilon} \sim \varepsilon^{\beta} \sim R^{-\beta},$$

which is where the stopping scale comes from.

3. **Comparison and conclusion** in Proposition 20.1.1: In the good gauge, we can interpolate and build an honest competitor, with controlled estimates. We use this to show that, the full (oriented) excess $\mathbf{E}$ is strongly approximated by the Dirichlet energy of some harmonic function, thus concluding the decay.

Part II

New weighted inequalities on two-manifolds

Chapter 3

Introduction

We provide L^2-weighted elliptic estimates for a class of positive weights $\omega \in W^{1,2}(\mathcal{M}^2)$ on smooth Riemannian connected two-manifolds $(\mathcal{M}^2, g)$ that weakly satisfy

$$\omega^2 \Delta_g \ln(\omega) = -\kappa(x)\omega^2, \tag{3.1}$$

with the weak formulation in Definition 4.0.1 and where Δ_g is the Laplace-Beltrami operator on $\mathcal{M}^2$. The original motivation of this article is to investigate the **weighted Hodge decomposition** of one-forms in two dimensions and provide estimates on the distance of the *weighted co-exact part* and the standard co-exact part as follows:

Lemma 3.0.1. *Let $(\mathcal{M}^2, g)$ be a Riemannian 2-manifold and let $\Omega \in \mathcal{M}^2$ be a smooth open domain and ω is a weight as in Definition 4.0.1 with $\kappa = 0$. Any smooth one-form $A \in C_c^\infty(\bigwedge^1 \Omega)$ has a Hodge decomposition and a weighted Hodge decomposition as follows:*

$$A = \star d\xi_1 + d\xi_2 \text{ and } \omega A = \star \omega d\phi_1 + \omega^{-1} d\phi_2\,,$$

for 4 compactly supported functions $\xi_1, \xi_2, \phi_1, \phi_2$. Moreover for any $0 \le \varepsilon \le C$ we have the estimates:

$$\|\omega^{1+\varepsilon} d(\xi_1 - \phi_1)\|^2_{L^2(\Omega)} \le C \frac{(\sup_\Omega \omega)^{2\varepsilon}}{\varepsilon^2} \|\omega^{-1} d\phi_2\|^2_{L^2(\Omega)} .$$

These estimates are instrumental in the quantitative stability of Yang-Mills-Higgs instantons in two dimensions in [45]. We present the results here in a more general setting, in the belief that these inequalities will be useful in other contexts.

In two dimensions, our results improve on Caffarelli-Kohn-Nirenberg inequalities [15] since we prove estimates for a wider class of weights, possibly vanishing on multiple points, with universal constant (e.g. $\omega = |x||x-1|$). There are also weights who satisfy (3.1) (e.g. $\omega = |x|$) which are not in any Muckenhoupt class.

3.1 Main results

Let $\Omega \subset \mathcal{M}^2$ be a smooth open connected domain and let λ_1 be the first Dirichlet eigenvalue of the Laplace-Beltrami operator on Ω. First we provide a generalization of Caffarelli-Kohn-Nirenberg interpolation inequalities in two dimensions:

Theorem 3.1.1. *Let $(\mathcal{M}^2, g)$ be a smooth 2-manifold and a weight ω as in Definition 4.0.1 and $\Omega \subset \mathcal{M}^2$ a smooth open domain. Then for any function $f \in C^\infty_c(\Omega)$ we have that:*

$$\int_\Omega |D\omega|^2 |f|^2 d\mathrm{vol}_g \le \int_\Omega \omega^2 |Df|^2 d\mathrm{vol}_g , \tag{3.1}$$

provided that $\kappa \le \lambda_1$.

In the next theorem we provide homogeneous elliptic estimates:

Theorem 3.1.2. *Let $(\mathcal{M}^2, g)$ be a smooth 2-manifold and a weight ω as in Definition 4.0.1 and*

$\Omega \subset \mathcal{M}^2$ a smooth open domain. Then for any function $f \in C_c^\infty(\Omega)$ we have that:

$$\int_\Omega \omega^2 |Df|^2 d\mathrm{vol}_g \leq \tau^{-1} \int_\Omega 2\frac{\omega^4}{|D\omega|^2}|\Delta_g f|^2 + 5|D\omega|^2|f|^2 d\mathrm{vol}_g\,, \tag{3.2}$$

provided that $-\frac{\lambda_1}{8}(2-\tau) \leq \kappa \leq \lambda_1$ for some $0 \leq \tau \leq 2$.

Theorem 3.1.3 is the main ingredient used in the proof of the Lemma 3.0.1 on the weighted Hodge decomposition. We break the homogeneity to remove the term $|D\omega|f$ from the right hand side, thereby introducing a constant on the right hand side as follows:

Theorem 3.1.3. *Let $(\mathcal{M}^2, g)$ be a smooth 2-manifold and a weight ω in Definition 4.0.1 with $\kappa = 0$ and $\varepsilon \geq 0$ and $\Omega \subset \mathcal{M}^2$ a smooth open domain. Then for any function $f \in C_c^\infty(\Omega)$ we have that:*

$$\int_\Omega \omega^{2+2\varepsilon}|Df|^2 d\mathrm{vol}_g \leq C\frac{(\sup_\Omega \omega)^{2\varepsilon}}{\varepsilon^2}\int_\Omega \frac{\omega^4}{|D\omega|^2}|\Delta_g f|^2 d\mathrm{vol}_g\,, \tag{3.3}$$

with the bound $C \leq \frac{8\varepsilon^2+5(1+\varepsilon)^4}{8(1+\varepsilon)^2}$ which is comparable to $\frac{5}{8}$ as $\varepsilon \to 0$.

Note that the Laplace-Beltrami operator Δ_g on functions $u \in W^{1,2}(\mathcal{M}, g)$ is defined by the duality relation below:

$$\int_\Omega -\Delta_g uv \; d\mathrm{vol}_g = \int_\Omega \langle Du, Dv\rangle \; d\mathrm{vol}_g\,, \text{ for all } v \in W_0^{1,2}(\Omega).$$

Examples of weights

In the case of $M := \mathbb{R}^2$ all weights of the form

$$\omega(x) = \Pi_{k=1}^n |x-a_k|^{\alpha_k} \text{ with } \{a_k\}_{k=1}^n \subset \mathbb{R}^2 \text{ and } \alpha_k > 0\,, \tag{3.4}$$

are admissible. More generally (as in [44]) for a smooth open and bounded domain $\Omega \subset M$ in a smooth two-manifold, the weights can take the following form:

$$\omega(x) = \Pi_{k=1}^{n} e^{-\alpha_k G_{p_k}(x)} \text{ with } \{p_k\}_{k=1}^{n} \subset \Omega \text{ and } \alpha_k > 0\,, \tag{3.5}$$

where $G_p(x) = G(p,x)$ is the Green's function for the domain Ω centered on p, namely the fundamental solution for the Laplacian on Ω (for a comprehensive account of the Green's function on smooth manifolds see [56]). Following the observation in [56, eq (1.1)], we see that there is some constant $C > 0$ such that any weight of the form (3.5) satisfies:

$$C^{-n}\omega(x) \leq \Pi_{k=1}^{n} d(x,p_k)^{\alpha_k} \leq C^{n}\omega(x) \text{ in } \Omega\,,$$

where $d(x,y)$ is the geodesic distance between x, y on M.

The weights (3.4) and (3.5) are generalizations of the Caffarelli-Kohn-Nirenberg interpolation results [15], in two dimensions. Moreover Theorem 3.1.1 and Theorem 3.1.3 provide weighted elliptic estimates for the weight $\omega = |x|^{\alpha}$:

$$\int_{\mathbb{R}^2} |x|^{2(\alpha-1)}|f|^2 \leq \alpha^{-2} \int_{\mathbb{R}^2} |x|^{2\alpha}|Df|^2\,,$$

$$\int_{\mathbb{R}^2} |x|^{2\alpha}|Df|^2 \leq \alpha^{-2} \int_{\mathbb{R}^2} |x|^{2(\alpha+2)}|\Delta f|^2 + \alpha^{2} \int_{\mathbb{R}^2} \frac{5}{2}|x|^{2(\alpha-1)}|f|^2\,,$$

$$\int_{B_1} |x|^{2(\alpha+\varepsilon)}|Df|^2 \leq C(\varepsilon\alpha)^{-2} \int_{B_1} |x|^{2(\alpha+1)}|\Delta f|^2\,,$$

provided that $\alpha > 0$.

The methods throughout the paper are inspired by [15] and [16] and are quiet elementary and only use Stokes theorem. A crucial part of our proof, equation (4.5), uses Lemma 4.0.2 which is an identity about symmetric matrices in two dimensions which does not hold in other dimensions.

Remark. In the case of unbounded domains (e.g. $\mathcal{M}^2 = \mathbb{R}^2$) we set $\lambda_1 = 0$ in Theorem 3.1.2 and Theorem 3.1.1.

Remark. Theorem 3.1.1 and Theorem 3.1.2 also work for the case of closed 2-manifolds $\Omega = \mathcal{M}^2$ with the assumption that $\int_\Omega \omega f \; d\mathrm{vol}_g = 0$. However Theorem 3.1.3 is a trivial statement for closed manifolds since the assumption $\kappa = 0$ tells us that $\Delta_g \omega^2 = 4|d\omega|^2 \geq 0$ and this means that the only admissible weights are constants.

Chapter 4

Proof of weighted inequalities

Definition 4.0.1. The weak formulation of (3.1) for a weight $\omega \in W^{1,2}(\mathcal{M}^2)$ is as follows: For any smooth test function $\phi \in C_c^\infty(\mathcal{M}^2)$ we have that:

$$\int_\Omega (4|D\omega|^2 - 2\kappa\omega^2)\phi - \omega^2\Delta_g\phi \; d\mathrm{vol}_g = 0\,.$$

To prove Theorem 3.1.1, Theorem 3.1.2 and Theorem 3.1.3 we use Stokes theorem to relate the integral of a carefully chosen positive term, to the difference of the right and the left hand side of (3.1) to (3.3).

Proof of Theorem 3.1.1. We begin with the identity below:

$$\begin{aligned} 0 \le \int_\Omega |D(\omega f)|^2 \; d\mathrm{vol}_g &= \int_\Omega |\omega Df + D\omega f|^2 \; d\mathrm{vol}_g \\ &= \int_\Omega \omega^2|Df|^2 + |D\omega|^2|f|^2 + 2\langle \omega D\omega, Dff\rangle \; d\mathrm{vol}_g\,. \end{aligned}$$

After completing the derivative for the cross term and using Definition 4.0.1 we see that:

$$\int_\Omega 2\langle \omega D\omega, Dff\rangle \; d\mathrm{vol}_g = \int_\Omega -\frac{\omega^2}{2}\Delta_g(f^2) \; d\mathrm{vol}_g = \int_\Omega (\kappa\omega^2 - 2|D\omega|^2)|f|^2 \; d\mathrm{vol}_g\,.$$

Then we use $\kappa \leq \lambda_1$ to estimate:

$$\int_\Omega \kappa\omega^2|f|^2 \, d\mathrm{vol}_g \leq \int_\Omega |D(\omega f)|^2 \, d\mathrm{vol}_g \, .$$

Finally we conclude that:

$$0 \leq \int_\Omega \omega^2|Df|^2 - |D\omega|^2|f|^2 \, d\mathrm{vol}_g \, .$$

□

Proof of Theorem 3.1.2. Similarly we begin by integrating a positive term:

$$0 \leq \int_\Omega \Big|\frac{\omega^2}{|D\omega|}\Delta_g f + |D\omega|f\Big|^2 \, d\mathrm{vol}_g = \int_\Omega \frac{\omega^4}{|D\omega|^2}|\Delta_g f|^2 + 2\omega^2 f\Delta_g f + |D\omega|^2|f|^2 \, d\mathrm{vol}_g \, .$$

By Stokes theorem for the cross term and Definition 4.0.1 we get that:

$$\int_\Omega 2\omega^2 f\Delta_g f \, d\mathrm{vol}_g = \int_\Omega -2\omega^2|Df|^2 + (4|D\omega|^2 - 2\kappa\omega^2)|f|^2 \, d\mathrm{vol}_g \, .$$

Since the assumption for an unbounded domain is $\kappa = 0$ the proof follows immediately. Otherwise by the assumption $-\kappa \leq \lambda_1(\frac{1}{4} - \frac{\tau}{8})$ we see that:

$$\int_\Omega -2\kappa|\omega f|^2 \, d\mathrm{vol}_g \leq \lambda_1(\frac{1}{2} - \frac{\tau}{4}) \int_\Omega |\omega f|^2 \, d\mathrm{vol}_g \, .$$

By the characterization of the first eigenvalue of the Laplace-Beltrami operator Δ_g we see that:

$$\lambda_1(\frac{1}{2} - \frac{\tau}{4}) \int_\Omega \omega^2|f|^2 \, d\mathrm{vol}_g \leq (\frac{1}{2} - \frac{\tau}{4}) \int_\Omega |D(\omega f)|^2 \, d\mathrm{vol}_g \, .$$

Since $\kappa \leq \lambda_1$, Theorem 3.1.1 applies and we get that:

$$(\frac{1}{2} - \frac{\tau}{4}) \int_\Omega |D(\omega f)|^2 \, d\mathrm{vol}_g \leq (2 - \tau) \int_\Omega \omega^2|Df|^2 \, d\mathrm{vol}_g \, .$$

Finally putting the estimates together, we conclude that:

$$0 \leq \int_{\Omega} 2\frac{\omega^4}{|D\omega|^2}|\Delta_g f|^2 + 5|D\omega|^2|f|^2 - \tau\omega^2|Df|^2 d\mathrm{vol}_g \,.$$

$\square$

In the proof of Theorem 3.1.3 we deal with the weighted hessian matrix $\omega^2 D^2 \ln(\omega)$ and by the condition (3.1) we know that it is a two dimensional symmetric trace-free matrix. The following lemma uses this structure and it is essential in the proof of Theorem 3.1.3:

Lemma 4.0.2. *Let $A \in \mathbb{R}^{2\times 2}$ be a symmetric matrix, namely $A^T = A$. Then we have that for any two real vectors $b, c \in \mathbb{R}^2$:*

$$2\langle A : b \otimes c\rangle\langle b, c\rangle - \langle A : b \otimes b\rangle|c|^2 - (A : c \otimes c)|b|^2 = trace(A)\langle b, c^{\perp}\rangle^2 \,, \tag{4.1}$$

where $\langle : \rangle$ is the matrix element-wise inner product and $c^{\perp}$ is the perpendicular vector to c.

Proof. We first calculate the expression above in dimension n. Since A is symmetric, it has n distinct perpendicular eigen-vectors e_i with real eigen-values μ_i. Then setting $b_i = \langle b, e_i\rangle$ and $c_i = \langle c, e_i\rangle$ we compute:

$$\begin{aligned} &2\langle A : b \otimes c\rangle\langle b, c\rangle - \langle A : b \otimes b\rangle|c|^2 - (A : c \otimes c)|b|^2 \\ &= \sum_{1\leq i,j\leq n} \mu_i (a_i c_j - c_i a_j)^2 \,. \end{aligned}$$

In the case $n = 2$:

$$2\langle A : b \otimes c\rangle\langle b, c\rangle - \langle A : b \otimes b\rangle|c|^2 - (A : c \otimes c)|b|^2 = \mathrm{trace}(A)(b_1 c_2 - c_1 b_2)^2 \,.$$

$\square$

Proof of Theorem 3.1.3. First we integrate a carefully chosen positive term of the form below:

$$
\begin{aligned}
0 &\leq \int_\Omega \Big|\frac{\omega^2}{|D\omega|}\Delta_g f + 2\omega\langle \frac{D\omega}{|D\omega|}, Df\rangle + 2|D\omega| f\Big|^2 \, d\mathrm{vol}_g \\
&= \int_\Omega \frac{\omega^4}{|D\omega|^2}|\Delta_g f|^2 + 4\omega^2\langle \frac{d\omega}{|D\omega|}, Df\rangle^2 + 4|D\omega|^2|f|^2 \qquad (4.2)\\
&+ 4\frac{\omega^3}{|D\omega|^2}\langle D\omega, Df\rangle \Delta_g f + 4\omega^2\Delta_g f f + 8\langle \omega D\omega, fDf\rangle \, d\mathrm{vol}_g \,. \qquad (4.3)
\end{aligned}
$$

Then for the first cross term in (4.3) we calculate by Stokes theorem and (3.1) (with the weak formulation in Definition 4.0.1) and the assumption $\kappa \geq 0$ that:

$$
\begin{aligned}
&\int_\Omega 4\frac{\omega^3}{|D\omega|^2}\langle D\omega, Df\rangle \Delta_g f \, d\mathrm{vol}_g \\
&= \int_\Omega 2\mathrm{div}_g(\frac{\omega^3}{|D\omega|^2} d\omega)|Df|^2 - 4D(\frac{\omega^3}{|D\omega|^2}D\omega) : Df \otimes Df \, d\mathrm{vol}_g \\
&\leq \int_\Omega (4\omega^2 - 4\frac{\omega^4}{|D\omega|^4}D^2(\log(\omega)) : D\omega \otimes D\omega)|Df|^2 \qquad (4.4)\\
&- 4\langle D(\frac{\omega^3}{|D\omega|^2}D\omega) : Df \otimes Df\rangle \, d\mathrm{vol}_g \,.
\end{aligned}
$$

The last line follows from the following:

$$
\begin{aligned}
2\mathrm{div}_g(\frac{\omega^3}{|D\omega|^2} d\omega) &= 2\frac{\omega^3}{|D\omega|^2}\Delta\omega + 6\omega^2 - 4\frac{\omega^3}{|D\omega|^4}D^2\omega : D\omega \otimes D\omega \\
&= 4\omega^2 + 2\frac{\omega^4}{|D\omega|^2}\Delta(\log(\omega)) - 4\frac{\omega^4}{|D\omega|^4}D^2(\log(\omega)) : D\omega \otimes D\omega \\
&\leq 4\omega^2 - 4\frac{\omega^4}{|D\omega|^4}D^2(\log(\omega)) : D\omega \otimes D\omega \,.
\end{aligned}
$$

Here we used the following identity:

$$
\omega D^2\omega = \omega^2 D^2 \ln(\omega) + d\omega \otimes d\omega \,,
$$

for the second term in (4.4). We get that:

$$
\begin{aligned}
&-4\langle D(\frac{\omega^3}{|D\omega|^2}D\omega) : Df \otimes Df\rangle - 4\frac{\omega^4}{|D\omega|^4}D^2(\log(\omega)) : D\omega \otimes D\omega |Df|^2 \\
=&-8\omega^2\langle Df, \frac{D\omega}{|D\omega|}\rangle^2 + 4\frac{\omega^4}{|D\omega|^4}\big[2\langle D^2\ln(\omega) : D\omega \otimes Df\rangle\langle D\omega, Df\rangle \\
&-\langle D^2\ln(\omega) : Df \otimes Df\rangle|D\omega|^2 - \langle D^2\ln(\omega) : D\omega \otimes D\omega\rangle|Df|^2\big]\,.
\end{aligned}
\tag{4.5}
$$

We apply Lemma 4.0.2 with:

$$
A = \omega^2 D^2\ln(\omega)\,, \quad b = \frac{D\omega}{|D\omega|} \quad \text{and } c = Df\,,
$$

and $\operatorname{trace}(A) = \omega^2\Delta_g\ln(\omega) = 0$ to see that:

$$
-4\langle D(\frac{\omega^3}{|D\omega|^2}D\omega) : Df \otimes Df\rangle - 4\frac{\omega^4}{|D\omega|^4}D^2(\log(\omega)) : D\omega \otimes D\omega|Df|^2 = -8\omega^2\langle Df, \frac{D\omega}{|D\omega|}\rangle^2\,.
$$

For the second and third cross term in (4.3) we see that:

$$
\int_\Omega 4\omega^2\Delta_g f f + 8\langle \omega D\omega, fDf\rangle \, d\mathrm{vol}_g = \int_\Omega -4\omega^2|Df|^2 \, d\mathrm{vol}_g\,.
$$

Then putting the estimates together we see that:

$$
4\int_\Omega \omega^2\langle Df, \frac{D\omega}{|D\omega|}\rangle^2 - |D\omega|^2|f|^2 \, d\mathrm{vol}_g \le \int_\Omega \frac{\omega^4}{|D\omega|^2}|\Delta_g f|^2 \, d\mathrm{vol}_g\,.
\tag{4.6}
$$

Using (3.1) with $\kappa = 0$ we get that for (4.6):

$$
\begin{aligned}
&\int_\Omega \omega^2 \langle Df, \frac{D\omega}{|D\omega|}\rangle^2 - |D\omega|^2 |f|^2 \, d\mathrm{vol}_g \\
=&\int_\Omega |\omega \langle Df, \frac{D\omega}{|D\omega|}\rangle + |D\omega| f|^2 \, d\mathrm{vol}_g \\
\geq&(\sup_\Omega \omega)^{-2\varepsilon} \int_\Omega \omega^{2\varepsilon} |\omega \langle Df, \frac{D\omega}{|D\omega|}\rangle + |D\omega| f|^2 \, d\mathrm{vol}_g \, .
\end{aligned}
\tag{4.7}
$$

Notice that $\omega^{1+\varepsilon}$ also satisfies (3.1) weakly in the case of $\kappa = 0$, so we compute (4.7) as follows:

$$
\begin{aligned}
&\int_\Omega \omega^{2\varepsilon} |\omega \langle Df, \frac{D\omega}{|D\omega|}\rangle + |D\omega| f|^2 \, d\mathrm{vol}_g \\
=&\int_\Omega \omega^{2+2\varepsilon} \langle Df, \frac{D\omega}{|D\omega|}\rangle^2 + \omega^{2\varepsilon}|D\omega|^2|f|^2 + 2\omega^{1+2\varepsilon}\langle D\omega, Df\rangle f \, d\mathrm{vol}_g \\
=&\int_\Omega \omega^{2+2\varepsilon} \langle Df, \frac{D\omega}{|D\omega|}\rangle^2 + \omega^{2\varepsilon}|D\omega|^2|f|^2 - \Delta_g(\frac{\omega^{2+2\varepsilon}}{2+2\varepsilon})|f|^2 \, d\mathrm{vol}_g \\
=&\int_\Omega \langle Df, \frac{D\omega}{|D\omega|}\rangle^2 - (1+2\varepsilon)\omega^{2\varepsilon}|D\omega|^2|f|^2 \, d\mathrm{vol}_g \, .
\end{aligned}
\tag{4.8}
$$

Notice that for $\omega^{1+\varepsilon}$ we have:

$$
\begin{aligned}
0 \leq&\int_\Omega \omega^{2\varepsilon} |\omega \langle Df, \frac{D\omega}{|D\omega|}\rangle + (1+\varepsilon)|D\omega| f|^2 \, d\mathrm{vol}_g \\
=&\int_\Omega \omega^{2+2\varepsilon} \langle Df, \frac{D\omega}{|D\omega|}\rangle + (1+\varepsilon)^2\omega^{2\varepsilon}|D\omega|^2|f|^2 + 2(1+\varepsilon)\omega^{1+2\varepsilon}\langle D\omega, Df\rangle f \, d\mathrm{vol}_g \\
=&\int_\Omega \omega^{2+2\varepsilon} \langle Df, \frac{D\omega}{|D\omega|}\rangle - (1+\varepsilon)^2\omega^{2\varepsilon}|D\omega|^2|f|^2 \, d\mathrm{vol}_g \, .
\end{aligned}
$$

We expand the square $(1+\varepsilon)^2$ to get a lower bound for (4.8):

$$
\int_\Omega \langle Df, \frac{D\omega}{|D\omega|}\rangle^2 - (1+2\varepsilon)\omega^{2\varepsilon}|D\omega|^2|f|^2 \, d\mathrm{vol}_g \geq \varepsilon^2 \int_\Omega \omega^{2\varepsilon}|D\omega|^2|f|^2 \, d\mathrm{vol}_g \, .
$$

and we get a preliminary inequality as follows:

$$\int_\Omega \omega^{2\varepsilon}|D\omega|^2|f|^2\, d\mathrm{vol}_g \le \frac{(\sup_\Omega \omega)^{2\varepsilon}}{4\varepsilon^2}\int_\Omega \frac{|\omega|^4}{|D\omega|^2}|\Delta_g f|^2\, d\mathrm{vol}_g\,. \tag{4.9}$$

Then we use Theorem 3.1.2 for $\omega^{1+\varepsilon}$ and $\kappa = 0$ and $\tau = 2$ to see that:

$$\int_\Omega 2\omega^{2+2\varepsilon}|Df|^2\, d\mathrm{vol}_g \le \int_\Omega 2\frac{\omega^{4+2\varepsilon}}{(1+\varepsilon)^2|D\omega|^2}|\Delta_g f|^2 + 5(1+\varepsilon)^2\omega^{2\varepsilon}|D\omega|^2|f|^2\, d\mathrm{vol}_g\,.$$

Finally we use (4.9) to conclude that:

$$\int_\Omega \omega^{2+2\varepsilon}|Df|^2\, d\mathrm{vol}_g \le (\frac{8\varepsilon^2+5(1+\varepsilon)^4}{8(1+\varepsilon)^2})\frac{(\sup_\Omega \omega)^{2\varepsilon}}{\varepsilon^2}\int_\Omega \frac{\omega^4}{|D\omega|^2}|\Delta_g f|^2\, d\mathrm{vol}_g\,.$$

□

Remark 4.0.3. In the case of $\mathcal{M}^2 = B_1^2(0) \subset \mathbb{R}^2$ and $\omega = |x|$ after the log-polar transformation $B_1^2 \to \mathbb{R}^+ \times S^1 = C$ by the map $t = -\ln(|x|)$ and $\theta = \arctan(\frac{y}{x})$ or equivalently a conformal change of metric with the factor $\frac{1}{|x|^2}$ and defining $f = |x|^{-1}u$ for $f \in C_1^\infty(B_1^2(0))$ we can see that:

$$\int_{B_1^2(0)} |D\omega|^2|f|^2 = \int_C |u|^2 d\mathrm{vol}_C\,, \tag{4.10}$$

$$\int_{B_1^2(0)} \omega^2|Df|^2 = \int_C |Du|^2 + |u|^2 d\mathrm{vol}_C\,,$$

$$\int_{B_1^2(0)} \frac{\omega^4}{|D\omega|^2}|Df|^2 = \int_C |\Delta u + 2\partial_t u + u|^2 d\mathrm{vol}_C\,. \tag{4.11}$$

After squaring and integrating by parts we see that (4.11) becomes:

$$\int_{B_1^2(0)} \frac{\omega^4}{|D\omega|^2}|Df|^2 = \int_C |\partial_{tt}u|^2 + |\partial_{t\theta}u|^2 + 2|\partial_t u|^2 + |\partial_{\theta\theta}u + u|^2\,.$$

We can see that if $u(t,\theta) = \sin(\theta)$ then (4.11) vanishes however (4.10) does not vanish so the term

$|D\omega| f$ on the right hand side of (3.2) is necessary. However the extra ε in the power

$$\int_{B_1^2(0)} \omega^{2+2\varepsilon} |Df|^2 = \int_C (|Du|^2 + |u|^2) e^{-2\varepsilon t} d\mathrm{vol}_C \,,$$

compactifies the domain $\mathbb{R}^+ \times S^1$ with a total measure of ε^{-2}. This provides some insight on Theorem 3.1.3 and the constants in (3.3).

We conclude the paper with the proof of the weighted Hodge decomposition estimates:

Proof of Lemma 3.0.1. We consider the two variational problems below:

$$\inf_{\xi \in C_c^\infty(\Omega)} \int_\Omega |A - \star d\xi|^2 \, d\mathrm{vol}_g \text{ and } \inf_{\phi \in C_c^\infty(\Omega)} \int_\Omega \omega^2 |A - \star d\phi|^2 \, d\mathrm{vol}_g \,. \tag{4.12}$$

Let $W_0^{1,2}(\omega^2, \Omega)$ be the completion of $C_c^\infty(\Omega)$ under the ω^2-weighted norm

$$\|u\|_{W_0^{1,2}(\omega^2,\Omega)} = \int_\Omega \omega^2 (|u|^2 + |du|^2) \,.$$

By Theorem 3.1.1 we see that

$$C^{-1} \|u\|_{W^{1,2}(\omega^2,\Omega)} \le \|\omega u\|_{W^{1,2}(\Omega)} \le C \|u\|_{W^{1,2}(\omega^2,\Omega)} \,,$$

and by the equivalence of the norms, the family of functions $\{u : \omega u \in W_0^{1,2}(\Omega)\}$ is equivalent to $W_0^{1,2}(\omega^2, \Omega)$ the existence of minimizers of (4.12) follows from convexity and the direct method in the calculus of variations. The Euler Lagrange equations for minimizers tell us that

$$\star d(A - \star d\xi_1) = 0 \Rightarrow \text{ there exists } \xi_2 \text{ such that } A - \star d\xi_1 = d\xi_2 \text{ and}$$

$$\star d(\omega^2 (A - \star d\phi_1)) = 0 \Rightarrow \text{ there exists } \phi_2 \text{ such that } \omega^2 (A - \star d\phi_1) = d\phi_2 \,.$$

in the sense of distributions. Then with a direct application of Theorem 3.1.3

$$\|\omega^{1+\varepsilon} d(\xi_1 - \phi_1)\|^2_{L^2(\mathcal{M}^2)} \leq C \frac{(\sup_{\mathcal{M}^2} \omega)^{2\varepsilon}}{\varepsilon^2} \|\frac{\omega^2}{|d\omega|} \Delta_g (\xi_1 - \phi_1)\|^2_{L^2(\mathcal{M}^2)}$$

and

$$\|\frac{\omega^2}{|d\omega|} \Delta_g (\xi_1 - \phi_1)\|^2_{L^2(\mathcal{M}^2)} = \|\frac{\omega^2}{|d\omega|} d(\omega^{-2} d\phi_2 - d\xi_1)\|^2_{L^2(\mathcal{M}^2)} = 4\|\omega^{-1} d\phi_2\|^2_{L^2(\mathcal{M}^2)} .$$

we conclude the proof. □

Part III

Quantitative stability of Yang-Mills-Higgs instantons in two dimensions

Chapter 5

Introduction

5.1 Background and main results

Let (u, ∇) be a section and connection on the trivial line bundle $\mathbb{R}^2 \times \mathbb{C} \to \mathbb{R}^2$. The *self-dual U(1)-Yang-Mills-Higgs* functional after a suitable re-scaling is

$$E(u, \nabla) = \int_{\mathbb{R}^2} |\nabla u|^2 + |F_\nabla|^2 + \frac{(1 - |u|^2)^2}{4}, \tag{5.1}$$

where F_∇ is the curvature two-form of ∇ (see Chapter 6 for details). One can show that in two dimensions the energy is lower bounded by a topological constant (see [10, 74])

$$E(u, \nabla) \geq 2\pi |N|, \tag{5.2}$$

where $N = \deg(u)$ is the rotation number of $\frac{u}{|u|}$ at infinity (Which is well defined when the energy (5.1) is finite, see Lemma 6.1.1). It is well known that minimizers of this functional satisfy a system of first order *vortex equations*, also known as the Bogomolny equations. In his PhD thesis ([74, 75]) C.H.Taubes shows that after prescribing the zero set, the solution *exists* and is *unique*, up to a change of gauge (see also [51]). Later in [62], A.Pigati and D.Stern consider the ε-rescaled

Yang-Mills-Higgs energy in higher dimensions:

$$E_\varepsilon(u,\nabla) = \int_{M^n} |\nabla u|^2 + \varepsilon^2 |F_\nabla|^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2},$$

where M^n is a compact Rimmannian n-manifold and they use this energy to construct minimal sub-manifolds of co-dimension two. Precisely, they show that in the $\varepsilon \to 0$ limit, the energy measure of critical points with uniformly bounded energy converge sub-sequentially to an integer rectifiable stationary varifold V of co-dimension two. Moreover they show that the currents dual to the curvature two-form converge to an integer rectifiable $(n-2)$-cycle Γ with $|\Gamma| \leq \mu_V$.

As a first step towards understanding the quantitative behavior of minimizers of the ε-rescaled energy in higher dimensions and regularity properties of the vortex set via a blow-up analysis, it is necessary to have a complete understanding of the stability of the energy (5.1) in two dimensions for arbitrary pairs. In fact, these estimates are mainly motivated by their importance in [28]. Roughly speaking, for an almost flat critical point of the re-scaled functional (5.3) in dimension $n \geq 3$, transversal 2-dimensional slices nearly minimize the two dimensional energy, with an error quantified by the *flatness.*

For any sharp functional inequality it is also natural to ask *"Can we estimate the distance to critical points by the discrepancy between the left and right hand side for some appropriate notion of distance?"*.

For instance this question has been extensively studied for the isoperimetric inequality (see [18, 19, 35–39, 58]) via methods of PDE and symmetrization. In [19] Cicalese and Leonardi use a penalization technique and regularity theory for quasi-minimizers of the perimeter to find uniform C^1 approximations of sets, for which they use PDE techniques to prove stability. The methods in this article are partly inspired from this approximation technique (see Chapter 8).

We first observe in Chapter 6 that the *discrepancy* is

$$E(u,\nabla) - 2\pi N = \int_{\mathbb{R}^2} r^2 |\star d\log(r) + A - d\theta|^2 + |\star dA - \frac{1-r^2}{2}|^2 ,$$

where $r = |u|$ and A is the real connection one-form of $\nabla : d - iA$. This leads us to investigate the *perturbed vortex equations*:

$$\star d\log|u| + A - d\theta = \frac{f_1}{|u|} \text{ and } \star dA - \frac{1-|u|^2}{2} = f_2 , \tag{5.3}$$

for some f_1, f_2 in $L^2(\mathbb{R}^2)$. The main difficulty is the error term $\frac{f}{|u|}$ for which Muckenhoupt theory [20] and Caffarelli-Kohn-Nirenberg inequalities [15] fall short. Łojasiewicz inequalities are also a possible technique (as used in classical GMT applications e.g. by L. Simon in [71]). However obtaining the inequality with the sharp power (such as the techniques of Topping in [78]) is rather difficult. However we are able to improve the existence and known results to the following sharp stability:

Theorem 5.1.1. *For any integer N there exists a constant $C_{|N|} > 0$ such that for any section and connection $(u,\nabla) \in W^{1,2}_{loc}(\mathbb{R}^2)$ on the trivial line bundle $L = \mathbb{R}^2 \times \mathbb{C} \to \mathbb{R}^2$ with $\deg(u) = N$ and small enough discrepancy $E(u,\nabla) - 2\pi|N|$ we have that:*

$$\min_{(u_0,\nabla_0)\in\mathcal{F}} \|u - u_0\|^2_{L^2(\mathbb{R}^2)} + \|F_\nabla - F_{\nabla_0}\|^2_{L^2(\mathbb{R}^2)} \leq C_{|N|} \left[E(u,\nabla) - 2\pi|N|\right] ,$$

where (up to a conjugation) $\mathcal{F}$ is the family of all N-vortex minimizers of the Yang-Mills-Higgs energy.

The proof relies on two main tools: Weighted elliptic inequalities of [44] (Recalled in Part II), in particular the weighted Hodge decomposition and a selection principle (inspired by [19]) using a penalized functional (see Chapter 8). Roughly speaking, this method is analogous to running the gradient flow for unit time. However the proof of existence for minimizers of the penalized functional (8.1) is straightforward as apposed to existence of the gradient flow (especially on an unbounded domain).

As a by product of these methods, we also prove a weighted Sobolev stability for *regular* enough pairs in the following theorem:

Theorem 5.1.2. *For any $\Lambda > 1$ and integer N, there exists constants $C_{\Lambda,|N|}, \eta_{\Lambda,|N|} > 0$ with the following property. Let $(u, \nabla) \in W^{1,2}_{loc}(\mathbb{R}^2)$ be an N-vortex section and connection such that*

(i) $\star d\left((\frac{u}{|u|})^*(d\theta)\right) = 2\pi \sum_{k=1}^{|N|} \delta_{x_k}$ *for a collection of points $\{x_k\}_{k=1}^{|N|} \subset \mathbb{R}^2$ counted with multiplicity.*

(ii) $E(u, \nabla) - 2\pi|N| \leq \eta^2_{\Lambda,|N|}$,

(iii) $\Lambda^{-1}|u_0| \leq |u| \leq \Lambda|u_0|$ *for some N-vortex solution (u_0, ∇_0) with $\{x_k\}_{k=1}^{|N|}$ as the zero set (counted with multiplicity).*

Then for any $0 < \varepsilon < \frac{1}{N}$

$$\int_{\mathbb{R}^2} |u_0|^{2+2\varepsilon} \left[\left| d \log\left(\frac{|u|}{|u_0|}\right)\right|^2 + |A_0 - A|^2 \right] \leq \frac{C_{\Lambda,|N|}}{\varepsilon^2} \left[E(u, \nabla) - 2\pi|N|\right] ,$$

up to a choice of $\pm A$.

Remark. The assumption *(i)* is satisfied if $\frac{u}{|u|} \in W^{1,1}_{loc}(\mathbb{R}^2)$. This is in fact a direct consequence of [52, Theroem 1.2]. However it is not clear if *(i)* can be inferred from *(ii)* and $(u, \nabla) \in W^{1,2}(\mathbb{R}^2)$.

A central tool in the analysis of the abelian Higgs model in any dimension n is the Yang-Mills-Higgs Jacobian $J(u, \nabla)$, which is a two-form defined as follows:

$$J(u, \nabla)(j, k) := \langle 2i\nabla_j u, \nabla_k u\rangle + \omega(j, k)(1 - |u|^2), \text{ for all } 1 \leq j < k \leq n,$$

where ω is the real curvature two-form associated to the connection ∇. It is the analogue of the Jacobian in the Ginzburg-Landau model (see [54]). In Section 7.3 using Theorem 5.1.2, we prove the second order stability of the Jacobian for *regular* pairs:

Theorem 5.1.3. *For any $\Lambda > 1$ and integer N there exists constants $C_{\Lambda,|N|}, \eta_{\Lambda,|N|} > 0$ such that for any $(u, \nabla) \in W^{1,2}_{loc}(\mathbb{R}^2)$ satisfying (i),(ii) and (iii) in Theorem 5.1.2 the following estimate holds:*

$$\int_{\mathbb{R}^2} |J(u, \nabla) - J(u_0, \nabla_0)| \leq C_{\Lambda,|N|} \sqrt{E(u, \nabla) - 2\pi|N|},$$

up to a conjugation of u.

To prove an improvement of flatness type result in [28], we rely on energy comparison with the pull-back of the two dimensional solution via a suitable harmonic competitor. The key difficulty there is to attach the boundary data which uses Theorem 5.1.1, Theorem 5.1.2 and Theorem 5.1.3 as the main ingredients (See [28, Proposition 10.2]).

For a non-optimal Łojasiewicz-type inequality for Yang-Mills-Higgs energy in complex geometry see [34, Section 1.3.3]. Related to the estimates in this paper, in [73, Section 3], D.Stuart defines a corrected Hessian for the Yang-Mills-Higgs energy (to mod out the gauge freedom) and derives coercive estimates. Further studies on the Yang-Mills-Higgs energy on Kähler manifolds has been done in [13, 46, 55]. For stability type results on the Yang-Mills functional we refer the reader to [76, 79]. It is also worthy to mention the articles [60, 62] for results connecting the variational theory of the re-scaled Yang-Mills-Higgs functional and minimal surfaces and [68] for the magnetic Ginzburg-Landau theory.

At the present we do not know how to obtain the estimates of Theorem 5.1.2 for the case $\varepsilon = 0$. The reason is that the embeddings of Lemma 3.0.1 are no longer compact for $\varepsilon = 0$. The stability in this problem is also deeply related to Poincaré inequalities on the unbounded one-sided cylinder $\mathbb{R}^+ \times S^1$ via the log-polar coordinate, which fail to be true. However by increasing the power in the weight by a factor of ε, analogously, assigning an exponentially decaying weight on the height

$e^{-\varepsilon t} : (t, \theta) \in \mathbb{R}^+ \times S^1 \to \mathbb{R}^+$, we are roughly compactifying the domain with a total measure of order ε^{-2}. This also serves as an intuition for the factor in the right hand side of Theorem 5.1.2. However the power 2 is sharp. We also state the following problem:

Open problem. *Do the estimates of Theorem 5.1.2 fail for the case $\varepsilon = 0$? In particular, does there exists a sequence of N-vortex pairs $\{(u_k, \nabla_k)\}_{k=1}^{\infty}$ such that the sharp stability fails in the following sense*

$$\lim_{k\to\infty} E(u_k, \nabla_k) = 2\pi|N| \text{ and } \lim_{k\to\infty} \frac{E(u_k, \nabla_k) - 2\pi|N|}{\min_{(u_0,\nabla_0)\in\mathcal{F}} \|du_k - du_0\|^2_{L^2(\mathbb{R}^2)}} = 0\,.$$

5.2 Results on compact surfaces

The Bogomolny trick also works on nontrivial line bundles $L \to M$ over closed surfaces. In this case the energy is lower bounded by the degree of the line bundle:

$$E(u, \nabla) \geq 2\pi|\deg(L)|\,. \tag{5.1}$$

In this case the *vortex equations* take the same form:

$$\nabla_{\partial_1} u + i\nabla_{\partial_2} u = 0 \text{ and } \star F_\nabla = \frac{1 - |u|^2}{2}\,. \tag{5.2}$$

Integrating the second equation over M, we see that:

$$|\deg(L)| \leq \frac{1}{4\pi}\mathrm{vol}(M)\,.$$

In [41] García-Prada obtained the analogues existence and uniqueness (for a slightly generalized functional) provided that the above constraint is satisfied. In this article we also prove the analogues stability, in the following theorem:

Theorem 5.2.1. *Let (M, g) be a smooth compact Riemann surface and $L \to M$ a Hermitian line bundle over M with $|\deg(L)| \leq \frac{1}{4\pi}\mathrm{vol}(M)$. Then there exists a constant $C_M > 0$ with the following property: Let (u, ∇) be a section and connection on L such that $E(u, \nabla) - 2\pi|\deg(L)|$ is small enough, then:*

$$\min_{(u_0,\nabla_0)\in\mathcal{F}} \|u - u_0\|^2_{L^2(M)} + \|F_\nabla - F_{\nabla_0}\|^2_{L^2(M)} \leq C_M \left[E(u, \nabla) - 2\pi|\deg(L)|\right] ,$$

where (up to a conjugation) $\mathcal{F}$ is the family all minimizers of the Yang-Mills-Higgs energy on $L \to M$.

Similarly for compact Riemann surfaces, we have the following result for *regular* pairs:

Theorem 5.2.2. *Let (M, g) be a smooth compact Riemann surface and $L \to M$ a Hermitian line bundle over M with $|\deg(L)| \leq \frac{1}{4\pi}\mathrm{vol}(M)$. For any $\Lambda > 1$, there exists $C_{\Lambda,M}, \eta_{\Lambda,M} > 0$ with the following property. Let $(u, \nabla) \in W^{1,2}_{loc}(M)$ be a pair such that*

(i) $\star d\left((\frac{u}{|u|})^*(d\theta)\right) = 2\pi\sum_{k=1}^{|\deg(L)|}\delta_{x_k}$ *for a collection of points* $\{x_k\}_{k=1}^{|\deg(L)|} \subset M$ *counted with multiplicity.*

(ii) $E(u, \nabla) - 2\pi|\deg(L)| \leq \eta^2_{\Lambda,M}$,

(iii) $\Lambda^{-1}|u_0| \leq |u| \leq \Lambda|u_0|$ *for some solution* (u_0, ∇_0) *with* $\{x_k\}_{k=1}^{|\deg(L)|}$ *as the zero set (counted with multiplicity).*

Then for any $0 < \varepsilon < \frac{1}{N}$:

$$\int_M |u_0|^{2+2\varepsilon}\left[\left|d\log\left(\frac{|u|}{|u_0|}\right)\right|^2 + |A_0 - A|^2\right] \leq \frac{C_{\Lambda,M}}{\varepsilon^2}\left[E(u, \nabla) - 2\pi|\deg(L)|\right] .$$

up to a conjugation of u. Moreover the following inequality holds for the Jacobian:

$$\int_M |J(u, \nabla) - J(u_0, \nabla_0)| \leq C_{\Lambda,M}\sqrt{E(u, \nabla) - 2\pi|\deg(L)|} .$$

5.3 Outline of the proof

Here we give an overview of the plan of Part III of the thesis:

Step 1. In Chapter 6 we prove that the degree is well defined for pairs with finite energy. Then we re-derive the first order vortex equations (6.4) and subsequently the PDE (6.1) and we define the *discrepancy* (6.1). Then in Proposition 6.2.1 we show that solutions are locally comparable to $\Pi_{k=1}^{M}|x-a_k|$.

Step 2. Here we explain the case of just one vortex for the sake of clarity. The ideas carry over to the case of multi vortex situations, since the elliptic and Poincaré type inequalities of [44] (recalled in Part II) have uniform and explicit constants. Now assume that $u = e^h u_0$ for a compactly supported real valued $h \in C_c^\infty(B_1(0))$ and a one-vortex solution u_0 centered at the origin. In this case after a suitable gauge transformation we can linearize the discrepancy $E(u,\nabla) - 2\pi = \eta^2$ as follows

$$\eta^2 \sim \int_{B_1} |x|^2 |\star dh + B|^2 + |\star dB + V(x)h|^2 \,,$$

where $h = \log\left(\frac{|u|}{|u_0|}\right)$ and $B = A - A_0$ and $C^{-1}|u_0|^2 \leq V(x) \leq C|u_0|^2$.

Now if the right hand side is zero, the first term tells us that $\star dh = -B$. Then we substitute this in the second term to see that $-\Delta h + V(x)h = 0$; testing this PDE with h and integrating by parts, we conclude that $h = 0$.

Now we aim to make this quantitative. First it is instructive to see the compactness argument if the first term was not weighted: Arguing by contradiction and scaling, we can assume that $\|h\|_{L^2} = 1$ while $\eta \to 0$. Then we see that $\star dh$ is uniformly close in L^2 topology to the co-exact part of $-B$, for which we have uniform $W^{1,2}$ bounds using the second term. Then we conclude that h (up to extracting a sub-sequence) converges to zero, strongly in L^2. The idea is to adapt this proof to the weighted case.

For this purpose we introduce a $|x|^2$-*weighted Hodge decomposition* of B as the minimizer of the following weighted functional

$$\inf_{v\in W^{1,2}_0(|x|^2,B_1)} \int_{B_1} |x|^2|B-\star dv|^2\,,$$

where $W^{1,2}(|x|^2,B_1)$ is the $|x|^2$-weighted Sobolev space (for details, see Part II). With the direct method in the calculus of variation and Theorem 3.1.1 we guarantee the existence of a minimizer. Then the Euler Lagrange equations of minimizers tells us that:

$$|x|B=\star|x|dv+|x|^{-1}df \text{ and } B=\star dp+dq\,.$$

Then we rewrite the linearized discrepancy using these identities. To be more precise we use the weighted Hodge decomposition for the first term and the standard Hodge decomposition for the second term:

$$\eta^2\sim\int_{B_1}|x|^2|d(h+v)|^2+|x|^{-2}|df|^2+|\Delta p+|u_0|^2h|^2\,. \tag{5.1}$$

Then we apply the weighted elliptic estimates of [44] recalled in Lemma 3.0.1 and we estimate the distance of the weighted and the standard Hodge decomposition as follows:

$$\int_{B_1}|x|^{2+2\varepsilon}|d(v-p)|^2\leq\frac{C}{\varepsilon^2}\int_{B_1}|x|^{-2}|df|^2\leq\frac{C}{\varepsilon^2}\eta^2\,,$$

for any $\varepsilon>0$. With this and (5.1) we get the uniform $W^{1,2}(B_1)$ estimates needed for $|x|^{1+\varepsilon}h$ near the vortex set to gain compactness.

To generalize this heuristic to many vortices, we note that all weights of the form $\Pi_{k=1}^M|x-a_k|^{\alpha_k}$ with $\alpha_k>0$ satisfy the condition Definition 4.0.1 to apply Theorem 3.1.1.

Then with a concentration compactness type argument we *glue* the local weighted estimates near the vortex set and uniform elliptic estimates far from the vortex set to conclude the stability for *regular* sections.

$\boxed{\textbf{Step 3.}}$ In Chapter 8 we generalize the stability to require only that the pair $(u, \nabla) \in W^{1,2}_{loc}(\mathbb{R}^2)$ nearly minimizes the energy. We use a *selection principle* and construct a new pair (u_1, ∇_1) by a finite iterative process of replacing (u, ∇) with the minimizer of the auxiliary functional

$$E(u_1, A_1) + \|u_1 - u\|^2_{L^2(\mathbb{R}^2)} + \|A_1 - A\|^2_{L^2(\mathbb{R}^2)} .$$

By Lemma 8.2.1 we gain some regularity at each step so after finitely many steps we have a pair $(\tilde{u}, \tilde{\nabla})$ with uniform $C^{N,\alpha}$ bounds in the local Coulomb gauge. Arguing by contradiction and Arzela-Ascoli we conclude that with small enough discrepancy u_1 is close enough to u_0 in C^N topology. Since u_0 is analytic, we see that u_1 is a C^N perturbation of a complex polynomial with degree $\leq N$. We then apply Lemma A.0.1 to see that C^N perturbations of complex polynomials with degree $M \leq N$ are uniformly comparable to another complex polynomial. This reduces the problem to Theorem 7.0.1 in Chapter 7.

$\boxed{\textbf{Step 4.}}$ We show that the methods above can be adapted, with little to no modification, to prove stability for nontrivial line bundles over arbitrary smooth compact Riemann surfaces. Here instead of polynomials, we use weights of the form $\Pi_{k=1}^{n} e^{-\alpha_k G_{x_k}(x)}$ with $\alpha_k > 0$, where $G_x(y)$ is the Green's function for a domain $\Omega \subset M$ in a Compact Riemann surface M. Note that in two dimensions, the Green's function for the Laplacian is proportional to $-\log d(x, y)$, where $d(x, y)$ is the geodesic distance between x, y on M; so essentially we work with weights proportional to $\Pi_{k=1}^{n} d(x, x_k)^{\alpha_k}$. Notice that estimates of Part II work with universal constants on any surface with boundary.

Chapter 6

The vortex equations

We work on Hermitian line bundles over smooth manifolds; on the trivial bundle $L = \mathbb{C} \times \mathbb{R}^2$, we can always write a metric connection ∇ as

$$\nabla = d - iA,$$

for a real-valued one-form, meaning that $\nabla_\xi s = ds(\xi) - i\alpha(\xi)s$.

In general, for two vector fields ξ and η, typically ∇_ξ and ∇_η do not commute, meaning that the connection has nontrivial *curvature*. Formally, the curvature F_∇ is given by

$$F_\nabla(\xi, \eta)(s) = [\nabla_\xi, \nabla_\eta]s - \nabla_{[\xi,\eta]}s. \tag{6.1}$$

A simple computation shows that F_∇ is a two-form with values in imaginary numbers; we will sometimes use the real-valued two-form ω given by

$$F_\nabla(\xi, \eta)(s) =: -i\omega(\xi, \eta)s. \tag{6.2}$$

On the trivial bundle, if $\nabla = d - iA$ then we simply have

$$\omega = dA.$$

Notice that the Yang-Mills-Higgs energy functional (5.1) enjoys the gauge invariance

$$(u, \nabla) \longrightarrow (ue^{i\xi}, \nabla - id\xi)\,,$$

for any compactly supported function $\xi \in C_c^\infty(\mathbb{R}^2)$. Moreover, after fixing the connection one-form $\nabla : d - iA$ we can rewrite the energy (5.1) as follows:

$$E(u, A) = \int_{\mathbb{R}^2} |du - iu \otimes A|^2 + |dA|^2 + \frac{(1 - |u|^2)^2}{4}\,.$$

The one-form A is sometimes called the *magnetic vector potential*. Observe that if $u = re^{i\theta}$ the energy can also be written in the following form:

$$E(re^{i\theta}, A) = \int_{\mathbb{R}^2} |dr|^2 + r^2|A - d\theta|^2 + |dA|^2 + \frac{(1 - r^2)^2}{4}\,. \tag{6.3}$$

Note that θ cannot be defined globally, however $d\theta$ is defined by pulling back the tangent vector to S^1 by the map $\frac{u}{|u|}$.

In [74] C.H.Taubes proves existence and uniqueness for minimizers of (5.1) using the **vortex equations** which are as follows (up to a conjugation or a change of orientation):

$$\star dr = -r(A - d\theta) \quad \star\, dA = \frac{1 - r^2}{2}\,. \tag{6.4}$$

6.1 The degree and the discrepancy for finite energy pairs

In the following lemma using the trick of Bogomolny [10] we first prove that pairs $(u, \nabla) \in W^{1,2}_{loc}(\mathbb{R}^2)$ with finite energy have globally well-defined degree. Moreover we also derive the *discrepancy* and the *vortex equations*.

Lemma 6.1.1. *Let $(u, \nabla) \in W^{1,2}_{loc}(\mathbb{R}^2)$ be a pair of section and connection on the trivial line bundle $\mathbb{C}\times\mathbb{R}^2$ with finite energy (5.1) $E(u, \nabla) = \Lambda < \infty$ and $u = re^{i\theta}$ (for some $\theta : \mathbb{R}^2 \to S^1$) and $\nabla : d - iA$. Then:*

(i) The degree of u, namely $\deg(u) = N$ is globally well defined.

(ii) The integral in (5.1) or equivalently (6.3) (possibly after a conjugation of u) can be rewritten as:

$$E(re^{i\theta}, A) = 2\pi|N| + \int_{\mathbb{R}^2} |\star dr + r(A - d\theta)|^2 + |\star dA - \frac{1 - r^2}{2}|^2 . \tag{6.1}$$

Proof. We use the notation $u = re^{i\theta}$ and $\nabla = d - iA$ as defined above. Note that $(u, \nabla) \in W^{1,2}_{loc}(\mathbb{R}^2) \subset \mathrm{VMO}_{loc}(\mathbb{R}^2)$, which is the space of functions with locally vanishing mean oscillation. We know from [14, II.2 Property 2] that the degree is locally well defined. We need to show that it is also globally well-defined. First name the sub-level set $Z_{1/2} = \{r < 1/2\}$ with the disjoint open and connected components $Z_{1/2} = \bigcup_{j=1}^{\infty} \Omega_j^{1/2}$. By the Coarea formula we have that:

$$\int_0^{1/2} \sum_{j=1}^{\infty} \mathcal{H}^1\big(\{r = t\} \cap \Omega_j^{1/2}\big)dt = \int_{Z_{1/2}} |dr| \leq C \int_{Z_{1/2}} |dr|^2 + \frac{(1 - r^2)^2}{4} \leq C\Lambda$$

Then by the mean value theorem we can find some threshold $\frac{1}{4} < \beta < \frac{1}{2}$ such that $\sum_{j=1}^{\infty} \mathcal{H}^1(\partial\Omega_j^{\beta}) < C\Lambda$. Here $Z_\beta = \bigcup_{j=1}^{\infty} \Omega_j^{\beta}$ are the disjoint connected components of $Z_\beta = \{|u| < \beta\}$. Now since perimeter bounds diameter we can see that:

$$\sum_{j=1}^{\infty} \operatorname{diam}(\Omega_j^\beta) < C\Lambda .$$

Since each Ω_j^β has finite diameter, we can see that the degree $\deg(u, \partial\Omega_j^\beta)$ is well defined on each domain. Now we proceed with a Cauchy-Schwartz and Stokes theorem:

$$\begin{aligned}
\infty > \Lambda &\geq \sum_{j=1}^{\infty} \int_{\Omega_j^\beta} |dr|^2 + r^2|A - d\theta|^2 + |dA|^2 + \frac{(1-r^2)^2}{4} \\
&\geq C \sum_{j=1}^{\infty} \left| \int_{\Omega_j^\beta} dA \frac{(\beta^2 - r^2)}{2} - r \star dr \wedge (A - d\theta) \right| \\
&= C \sum_{j=1}^{\infty} \left| \int_{\Omega_j^\beta} dA \frac{(\beta^2 - r^2)}{2} + \star d(\beta^2 - r^2) \wedge (A - d\theta) \right| \\
&= C \sum_{j=1}^{\infty} \left| \int_{\Omega_j^\beta} d(d\theta)(\beta^2 - r^2) \right| = C\beta^2 \sum_{j=1}^{\infty} |\deg(u, \partial\Omega_j^\beta)|
\end{aligned}$$

The last line follows from $\int_{\Omega_j^\beta} d(d\theta) = \int_{\partial\Omega_j^\beta} \partial_\tau \theta = \deg(u, \Omega_j^\beta)$. This tells us that only finitely many of the domains Ω_j^β have non-zero degree. Hence we can see that the degree is also globally well-defined. We also name

$$\deg(u) = N .$$

To prove the second point, we again replace $|dr|^2$ with $|\star dr|^2$ and complete the squares in (6.3):

$$\begin{aligned}
&E(re^{i\theta}, A) = \\
&\quad = \int_{\mathbb{R}^2} |\star dr|^2 + r^2|A - d\theta|^2 + |dA|^2 + \frac{(1-r^2)^2}{4} \\
&\quad = \int_{\mathbb{R}^2} |\star dr + r(A - d\theta)|^2 + |\star dA - \frac{1-r^2}{2}|^2 - 2r dr \wedge (A - d\theta) + dA(1 - r^2) .
\end{aligned}$$

Using $d(1-r^2) = -2rdr$ and Stokes theorem and noticing that

$$\int_{\mathbb{R}^2} -2rdr \wedge (A - d\theta) + dA(1-r^2) \leq CE(u, \nabla) < \infty\,,$$

we see that:

$$E(re^{i\theta}, A) = \int_{\mathbb{R}^2} |\star dr + r(A - d\theta)|^2 + |\star dA - \frac{1-r^2}{2}|^2 + d(d\theta)(1-r^2)\,.$$

For any compact set $K \in \mathbb{R}^2 \backslash \{r = 0\}$ we see that $d\theta \in W^{1,2}(K)$ hence $d(d\theta)$ is supported on the zero set of r we can see that:

$$\int_{\mathbb{R}^2} d(d\theta)(1-r^2) = \sum_{j=1}^{\infty} \int_{\Omega_j^\beta} d(d\theta)(1-r^2) = \sum_{j=1}^{\infty} \int_{\Omega_j^\beta} d(d\theta) = 2\pi \deg(u)\,.$$

Then after a possible conjugation of u we see that

$$E(re^{i\theta}, A) = 2\pi|\deg(u)| + \int_{\mathbb{R}^2} |\star dr + r(A - d\theta)|^2 + |\star dA - \frac{1-r^2}{2}|^2\,.$$

We get equality if and only if:

$$\star dr = -r(A - d\theta) \qquad \star dA = \frac{1-r^2}{2}\,.$$

These are called the first order *vortex equations*. □

Remark 6.1.2. Assuming sufficient decay for $|\nabla u|(x)$ as $|x| \to \infty$ (see the conditions in [51, Chapter II, Theroem 3.1]) we can also see that:

$$\int_{\mathbb{R}^2} dA = 2\pi N\,.$$

For a detailed discussion see [51, Chapter II.3]. But here we do not need this result. Notice that even if dA is sufficiently close to an integral two-form in $L^2(\mathbb{R}^2)$, since $\mathbb{R}^2$ is unbounded, one can draw no conclusions on the integrality of dA.

6.2 Estimates on solutions of the vortex equations

In the sequel without loss of generality, we assume that $N \geq 0$ after a possible conjugation of u. Here we first prove that solutions of the vortex equations are comparable to modulus of polynomials on their sublevel sets. We collect these information in the following lemma:

Proposition 6.2.1. *There exists a constant $C_N > 1$ depending only on $N \geq 0$ with the following property: Let (u_0, ∇_0) be a solution to the vortex equations (6.4) as in [74] with the prescribed zero set $x_1, \dots, x_N \in \mathbb{R}^2$ counted with multiplicity. Then there exists $M \leq N$ balls $\{B_{\rho_k}(z_k)\}_{k=1}^M$ and some $\frac{1}{4} \leq \beta \leq \frac{1}{2}$ such that:*

(i) $Z_\beta = \{|u_0| \leq \beta\} \subset \bigcup_{k=1}^M B_{\rho_k}(z_k)$,

(ii) $B_{2\rho_i}(z_i) \cap B_{2\rho_j}(z_j) = \emptyset$ *for all* $1 \leq i < j \leq M$,

(iii) $1 \leq \rho_k \leq C_N$ *for all* $1 \leq k \leq M$,

(iv) $C_N^{-1}\omega_k \leq |u_0| \leq C_N \omega_k$ *in* $B_{2\rho_k}(z_k)$, *s.t.* $\omega_k(x) = \Pi_{x_i \in B_{\rho_k}(z_k)} |x - x_i|$ *for all* $1 \leq k \leq M$.

Proof. First we estimate the measure of $\{|u| \leq \frac{1}{2}\}$. Then we use the co-area formula and a mean value theorem to find a sub-level set $\{|u| \leq \beta\}$ set with bounded total perimeter. Then we cover them with non intersecting balls and use (6.1) and the maximum principle.

Name $r_0 = |u_0|$; by the *vortex equations*, we know that r_0 is a solution of the following

$$-\Delta \log(r_0) + \frac{1}{2}(r_0^2 - 1) = -\sum_{i=1}^N 2\pi \delta_{x_i}\,, \tag{6.1}$$

where $\{x_1, \dots, x_N\} \subset \mathbb{R}^2$ is the zero set of r_0 (counted with multiplicity) and δ_x is a point-mass on $x \in \mathbb{R}^2$. Define the sub-level set and its disjoint connected components $Z_\beta = \{r_0 \leq \beta\} = \bigcup_{j=1}^\infty Z_\beta^j$.

Then we multiply (6.1) by $r_0^2 - \beta^2$ and integrate by parts on each Z_β^j:

$$0 \leq \int_{Z_\beta^j} 2|dr_0|^2 + \frac{1}{2}(1 - r_0^2)(\beta^2 - r_0^2) = 2\pi\beta^2 K_j \,.$$

Here K_j is the number of zeros in Z_β^j; if $K_j = 0$ we see that $r_0 = \beta$ on Z_β^j and since solutions of the vortex equations are analytic (by [74, Proposition 6.1]), unique continuation tells us that $r_0 = \beta$ globally, which is a contradiction if $\beta < 1$ or $N \neq 0$; so we conclude that $K_j \geq 1$ for finitely many js. This means that there are at most N connected components of Z_β for all $\beta < 1$.

Now we sum the estimates on $Z_{\frac{3}{4}}$:

$$|Z_{\frac{1}{2}}| + \int_{Z_{\frac{1}{2}}} |dr_0|^2 \leq C \int_{Z_{\frac{3}{4}}} 2|dr_0|^2 + \frac{1}{2}(1 - r_0^2)(\frac{9}{16} - r_0^2) \leq C_N \,.$$

By the co-area formula and mean-value theorem we get that there exists some $\frac{1}{4} \leq \beta \leq \frac{1}{2}$ such that:

$$\mathcal{H}^1(\partial Z_\beta) \leq C \int_{\frac{1}{4}}^{\frac{1}{2}} \mathcal{H}^1(\partial Z_s) ds \leq C \int_{Z_{\frac{1}{2}}} |dr_0| \leq C |Z_{\frac{1}{2}}|^{\frac{1}{2}} \Big(\int_{Z_{\frac{1}{2}}} |dr_0|^2 \Big)^{\frac{1}{2}} \leq C_N \,.$$

Since the set Z_β has at most N connected components $\{Z_\beta^j\}_{j=1}^M$ for some $M \leq N$ we get that $\max_{1 \leq j \leq M} \operatorname{diam}(Z_\beta^j) \leq C_N$. Then we find M balls $B_r(z_1), \ldots, B_r(z_M)$ with $1 \leq r \leq C_N$ whose union covers $Z_\beta \subset \cup_{i=1}^M B_r(z_i)$ and $z_i \in Z_\beta$. To find the balls covering Z_β satisfying the assumptions, at each step we replace any two balls $B_{r_i}(z_i), B_{r_j}(z_j)$ that, if dilated, intersect $B_{2r_i}(z_i) \cap B_{2r_j}(z_j) \neq \emptyset$ with the ball $B_{3(r_i + r_j)}(\frac{z_i + z_j}{2})$. Since at each step the number of balls decreases, this procedure stops at maximum $N - 1$ steps. Notice that in each step $\max(r_i)$ increases at most by a factor of 6 so we are left with $M \leq N$ balls $B_{\rho_1}(z_i), \ldots, B_{\rho_M}(z_M)$ such that $1 \leq \rho_i \leq C_N$.

Now for each $1 \leq j \leq M$ define the weight $\omega_j = \Pi_{i=1}^{K_j} |x - x_i|$ where $x_1, \ldots, x_{K_j}$ are the zeros of r_0 in the ball $B^j_{\rho_j}(z_j)$. Then we estimate $\|\log(\omega_j) - \log(r_0)\|_{L^\infty(B^j_{2\rho}(z_j))}$ using (6.1):

$$-\Delta h = \frac{1}{2}(1 - r_0^2) \text{ in } B^j_{2\rho_j}(z_j)\,,$$
$$h = \log(r_0) - \log(\omega_j)\,.$$

Notice that by (6.1) we have $\Delta r_0^2 = 4|dr_0|^2 - r_0^2(1 - r_0^2)$ and with an application of the maximum principle and $\int_{\mathbb{R}^2} |dr_0|^2 < \infty$ we can see that $r_0 \leq 1$. We then notice that:

$$-\Delta h \geq 0 \text{ and } -\Delta(h - \rho_j^2 + \frac{1}{4}|x - z_j|^2) \leq 0 \text{ in } B^j_{2\rho_j}(z_j)\,.$$

By the weak maximum principle we get the estimates below:

$$\|h\|_{L^\infty(B^j_{2\rho_j}(z_j))} \leq \rho_j^2 + \sup_{\partial B^j_{2\rho_j}(z_j)} |h| \leq C_N^2 + \sup_{\partial B^j_{2\rho_j}(z_j)} |\log(r_0)| + |\log(\omega_j)|\,.$$

From the bound on $\beta \geq \frac{1}{4}$ we know that $\partial B^j_{2\rho_j}(z_j) \subset Z^c_\beta \subset \{r_0 > \frac{1}{4}\}$ and this yields an upper bound on $|\log(r_0)|$ on the boundary of the ball. Since $\{x_1, \ldots, x_{K_j}\} \in B^j_{\rho_j}(z_j)$ and $\rho_j \geq 1$, we get a lower bound for ω_j which combined with $\rho_j \leq C_N$ gives us an upper bound on $|\log(\omega_j)|$. Finally we estimate:

$$\|h\|_{L^\infty(B^j_{2\rho_j}(z_j))} \leq C_N \Rightarrow e^{-C_N}\omega_j \leq r_0 \leq e^{C_N}\omega_j \text{ in } B^j_{2\rho_j}(z_j)\,,$$

and conclude. □

Chapter 7

Stability of regular pairs

In this section we show stability under some regularity conditions:

Theorem 7.0.1. *For any $\Lambda > 1$ and $N \in \mathbb{N}$ there exists $\eta_{\Lambda,N}, C_{\Lambda,N} > 0$ with the following property. Let (u, ∇) be a section and connection on the trivial line bundle $\mathbb{C} \times \mathbb{R}^2 \to \mathbb{R}^2$ that satisfy:*

(i) $\star d\left((\frac{u}{|u|})^*(d\theta)\right) = 2\pi \sum_{k=1}^{N} \delta_{x_k}$ *for a collection of points $\{x_k\}_{k=1}^{|N|} \subset \mathbb{R}^2$ counted with multiplicity.*

(ii) $E(u, \nabla) - 2\pi N \leq \eta_{\Lambda,N}^2$,

(iii) $\Lambda^{-1}|u_0| \leq |u| \leq \Lambda|u_0|$ *for some N-vortex solution (u_0, ∇_0) with $\{x_k\}_{k=1}^{|N|}$ as the zero set (counted with multiplicity).*

Then:

$$\| |u| - |u_0| \|_{L^2(\mathbb{R}^2)}^2 + \|F_\nabla - F_{\nabla_0}\|_{L^2(\mathbb{R}^2)}^2 \leq C_{\Lambda,N} \left[E(u, \nabla) - 2\pi N\right] .$$

We divide the proof of this theorem to two parts. In the first part we deal with estimates near the vortex set and in the second part we combine these estimates with uniform elliptic estimates far from the vortex set.

7.1 Compactness estimates near the vortex set

This section contains the main ingredients of the proof, which are weighted inequalities near the vortex set, in the following proposition:

Proposition 7.1.1. *For any constant* $\Lambda > 0$*, radius* $R > 0$ *and integer* $N \in \mathbb{N}$ *there exists a constant* $C_{\Lambda,R,N} > 0$ *with the following property. For any function* $h \in C_c^\infty(B_R)$*, one-form* $B \in C_c^\infty(\bigwedge^1 B_R)$ *and weight* ω *such that*

$$\omega(x) = \Pi_{i=1}^M |x - x_i| \text{ for } x_1, \ldots, x_M \in B_R \text{ counted with multiplicity for } 1 \leq M \leq N\,,$$

we have the following inequality:

$$\int_{B_R} \omega^2 |h|^2 \leq C_{\Lambda,R,N} \int_{B_R} \omega^2 |\star dh + B|^2 + |\star dB + V(x)h|^2\,,$$

provided that $0 \leq V(x) \leq \Lambda \omega(x)^{1+\frac{1}{N}}$.

Proof. Here we implicitly use the fact that all positive powers of ω as above are admissible weights for Definition 4.0.1. Note that for any weight $\omega(x) = \Pi_{i=1}^M |x - x_i|^{\alpha_i}$ with $\alpha_i > 0$ the condition Definition 4.0.1 is satisfied. For any $\phi \in C_c^\infty(\mathbb{R}^2)$

$$\begin{aligned}
\int_{\mathbb{R}^2} \omega^2 \Delta \log(\omega) \phi &= \int_{\mathbb{R}^2} \omega(x) \sum_{i=1}^M \alpha_i \Delta \log |x - x_i| \phi \\
&= 2\pi \int_{\mathbb{R}^2} \omega(x) \sum_{i=1}^M \alpha_i \delta_{x_i} \phi = 2\pi \sum_{i=1}^M \alpha_i \omega(x_i) \phi(x_i) = 0\,.
\end{aligned}$$

The last line follows from $\omega(x_i) = 0$ if $\alpha_i > 0$.

Now we divide the rest of the proof into 5 steps:

$\boxed{\textit{Step 1.}}$ We argue by contradiction. Assume there is a sequence $\{h_k, B_k, \omega_k, V_k\}_{k=1}^\infty$ satisfying the assumptions such that:

$$\int_{B_R} \omega_k^2 |h_k|^2 = 1\,,$$

$$\delta_k^2 = \int_{B_R} \omega_k^2 | \star dh_k + B_k|^2 + | \star dB_k + V_k(x) h_k|^2 \longrightarrow 0 \text{ as } k \longrightarrow \infty\,.$$

Now by Lemma 3.0.1 we write the standard Hodge decomposition and the *weighted Hodge decomposition* for the one-form B_k:

$$B_k = \star dp_k + dq_k \text{ and } \omega_k B_k = \star \omega_k dv_k + \omega_k^{-1} df_k\,, \tag{7.1}$$

where $\omega_k^{-1} df_k \in L^2(\mathbb{R}^2)$, $q_k, \omega_k v_k \in W_0^{1,2}(B_R)$ and $p_k \in W_0^{2,2}(B_R)$; since $\Delta p_k = dB_k \in L^2(B_R)$.

Now we rewrite δ_k^2 using (7.1) and Stokes theorem to see that:

$$\delta_k^2 = \int_{B_R} \omega_k^2 |d(h_k + v_k)|^2 + \omega_k^{-2} |df_k|^2 + |\Delta p_k + V_k(x) h_k|^2\,. \tag{7.2}$$

Notice that the term dq_k has disappeared from the expression and this is consistent with the gauge invariance.

By *(ii)* in Lemma 3.0.1 for any $\varepsilon > 0$ we estimate:

$$\int_{B_R} \omega_k^{2+2\varepsilon} |d(v_k - p_k)|^2 \leq C_{R,N} \varepsilon^{-2} \int_{B_R} \omega_k^{-2} |df_k|^2 \leq C_{R,N} \frac{\delta_k^2}{\varepsilon^2}\,. \tag{7.3}$$

Our goal is to use (7.2) to find uniform $W^{1,2}$ upper bounds on $\omega_k^{1+\varepsilon} v_k$ (for small enough $\varepsilon > 0$) and a uniform non-zero lower bound on its L^2 norm to arrive at a contradiction.

$\boxed{\textit{Step 2.}}$ In this step we find a lower bound for the L^2 norm of $\omega^{1+\frac{1}{2N}} h_k$. First we show that there exists a positive constant $C_{R,N} > 0$ such that for any $\varepsilon < \frac{1}{N}$ we have the point-wise bound:

$$\omega_k^2 \leq C_{R,N} \left(\omega_k^{2+2\varepsilon} + |d\omega_k|^2 \omega_k^{2\varepsilon} \right) \Leftrightarrow \omega_k^{2\varepsilon} \left(1 + |d \log(\omega_k)|^2 \right) \geq C_{R,N}\,, \tag{7.4}$$

for weights ω_k as in the statement of this proposition. Arguing by contradiction, since ω_ks form a compact family, (7.4) fails if and only if for some $\{y_j\}_{j=1}^{\infty} \in B_R$ we have:

$$\lim_{j\to\infty} \omega_k^{2\varepsilon}(y_j)\left(1 + |d\log\omega_k(y_j)|^2\right) = 0\,.$$

We can see by compactness that there exists some $y \in B_R$ such that $\omega_k^{2\varepsilon}(y)(1+|d\log\omega_k(y)|^2) = 0$, meaning also that $\omega_k(y) = 0$; now since the vanishing order at zeros of ω_k^{ε} with at most N roots is less than $\varepsilon N < 1$, we can see that ω_k^{ε} vanishes slower than $|d\log(\omega_k)| \sim O(|x - |^{-1})$; hence (7.4) follows. Then we can estimate:

$$1 = \int_{B_R} \omega_k^2|h_k|^2 \le C_{R,N}\int_{B_R}\left(\omega_k^{2+2\varepsilon} + |d\omega_k|^2\omega_k^{2\varepsilon}\right)|h_k|^2 \le C_{R,N}\int_{B_R}\omega_k^{2+2\varepsilon}|dh_k|^2\,.$$

In the last inequality we used Theorem 3.1.1 with the weight $\omega_k^{1+\varepsilon}$.

In the rest of the proof we fix

$$\varepsilon = \frac{1}{2N}\,.$$

$\boxed{\textit{Step 3.}}$ In this step we show that $\omega^{1+\varepsilon}v_k$ is also lower bounded in L^2. We use (7.3) to see:

$$\begin{aligned}\int_{B_R}\omega_k^{2+2\varepsilon}|dh_k|^2 &\le C_{R,N}\left(\delta_k^2 + \int_{B_R}\omega_k^{2+2\varepsilon}|dv_k|^2\right)\\ &\le C_{R,N}\left(\frac{\delta_k^2}{\varepsilon^2} + \int_{B_R}\omega_k^{2+2\varepsilon}|dp_k|^2\right)\,.\end{aligned}\tag{7.5}$$

Now by standard elliptic estimates we have that for $p \in W_0^{2,2}(B_R)$:

$$\begin{aligned}\int_{B_R}\omega_k^{2+2\varepsilon}|dp_k|^2 &\le C_{R,N}\int_{B_R}|dp_k|^2\\ \le C_{R,N}\int_{B_R}|\Delta p_k|^2 &\le C_{R,N}\left(\delta_k^2 + \int_{B_R}V_k^2|h_k|^2\right)\,.\end{aligned}\tag{7.6}$$

Then by the point-wise bound $V(x) \leq \Lambda \omega_k^{1+\frac{1}{N}}$ we get that:

$$\int_{B_R} V_k^2 |h_k|^2 \leq C_{R,N,\Lambda} \int_{B_R} \omega_k^{2+2\varepsilon} |h_k|^2 \leq C_{R,N,\Lambda} \int_{B_R} \omega_k^{2+2\varepsilon} \left(|v_k|^2 + |v_k + h_k|^2 \right) .$$

By Poincaré inequality and Theorem 3.1.1 with the weight $\omega_k^{1+\varepsilon}$, we get that:

$$\int_{B_R} \omega_k^{2+2\varepsilon} |v_k + h_k|^2 \leq C_R \int_{B_R} \left| d \left(\omega_k^{1+\varepsilon} (v_k + h_k) \right) \right|^2 \leq C_{R,N,\Lambda} \delta_k^2 .$$

Noting that $\varepsilon = \frac{1}{2N}$, we get the following lower bound:

$$C_{R,N,\Lambda} \leq \int_{B_R} \omega_k^{2+2\varepsilon} |v_k|^2 , \tag{7.7}$$

provided that k is large enough.

$\boxed{\textit{Step 4.}}$ In this step we find uniform upper bounds on the $W^{1,2}(B_R)$ norm of $\omega^{1+\frac{1}{2N}} v_k$. For this we apply Theorem 3.1.1 with $\omega_k^{1+\varepsilon}$, the inequality (7.3) and standard elliptic estimates as follows:

$$\begin{aligned} \|\omega_k^{1+\varepsilon} v_k\|^2_{W_0^{1,2}(B_R)} &\leq C_R \int_{B_R} |\omega_k|^{2+2\varepsilon} |dv_k|^2 \leq C_{R,N} \left(\delta_k^2 + \int_{B_R} \omega_k^{2+2\varepsilon} |dp_k|^2 \right) \\ &\leq C_{R,N} \left(\delta_k^2 + \int_{B_R} |\Delta p_k|^2 \right) \leq C_{R,N} \left(\delta_k^2 + \int_{B_R} V_k^2 |h_k|^2 \right) \\ &\leq C_{\Lambda,R,N} \left(\delta_k^2 + \int_{B_R} \omega_k^2 |h_k|^2 \right) \leq C_{\Lambda,R,N} . \end{aligned}$$

We also bound the weighted Sobolev norms of h_k, p_k and f_k:

$$\|\omega_k^{-1} df_k\|^2_{L^2(B_R)} \leq \delta_k^2 \, , \ \|p_k\|^2_{W_0^{2,2}(B_R)} \leq C_{R,N,\Lambda} \text{ and } \|\omega_k h_k\|^2_{L^2(B_R)} \leq C_{R,N,\Lambda} .$$

Using Theorem 3.1.1 with ω_k we also estimate $\omega_k v_k$:

$$\|\omega_k v_k\|^2_{L^2(B_R)} \leq 2\int_{B_R} \omega_k^2 \left(|h_k|^2 + |v_k + h_k|^2\right) \leq C_R \left(1 + \int_{B_R} |d\left(\omega_k \left(h_k + v_k\right)\right)|^2\right)$$
$$\leq C_R \left(1 + \int_{B_R} \omega_k^2 |dh_k + dv_k|^2\right) \leq C_R \left(1 + \delta_k^2\right) \leq C_R\,.$$

$\boxed{\textit{Step 5.}}$ In this last step we finish the compactness argument. By the compact embedding $W^{1,2}_0(B_R) \hookrightarrow_c L^2(B_R)$, Banach–Alaoglu and Rellich–Kondrachov theorem we can extract a sub-sequence k_j with some $g_\infty, f_\infty, p_\infty, \omega_\infty$ such that:

$$\begin{cases} \omega_{k_j}^{1+\varepsilon} v_{k_j} \longrightarrow \omega_\infty^{1+\varepsilon} v_\infty & \text{strongly in } L^2(B_R)\,, \\ p_{k_j} \longrightarrow p_\infty & \text{strongly in } W^{1,2}_0(B_R)\,, \\ \omega_{k_j} v_{k_j} \rightharpoonup \omega_\infty v_\infty & \text{weakly in } L^2(B_R)\,, \\ \omega_{k_j}^{-1} df_{k_j} \rightharpoonup \omega_\infty^{-1} df_\infty & \text{weakly in } L^2(B_R)\,, \\ \omega_{k_j} h_{k_j} \rightharpoonup \omega_\infty h_\infty & \text{weakly in } L^2(B_R)\,, \\ \omega_{k_j} \longrightarrow \omega_\infty & \text{in } C^k(B_R) \text{ for all } k \geq 0\,, \\ V_{k_j} \rightharpoonup V_\infty & \text{weakly, in duality with } L^1(B_R)\,. \end{cases}$$

Since $\delta_{k_j} \longrightarrow 0$ vanishes in the limit. By lower semi-continuity and (7.3)

$$f_\infty = 0 \text{ and } p_\infty = g_\infty \Rightarrow -\Delta v_\infty + V_\infty(x) v_\infty = 0 \text{ in the sense of distributions}\,.$$

Finally we test by v_∞ and use $V_\infty(x) \geq 0$ to get that $dv_\infty = 0$. Since v vanishes on the boundary then $v_\infty = 0$, but this contradicts (7.7) which concludes the proof. □

Corollary 7.1.2. *As a consequence of the estimates in Proposition 7.1.1 we also get that there exists* *a constant* $C_{\Lambda,R,N}$ *such that for any* $0 < \varepsilon < \frac{1}{N}$*:*

$$\int_{B_R} \omega^{2+2\varepsilon} |dh|^2 \leq \frac{C_{\Lambda,R,N}}{\varepsilon^2} \int_{B_R} \omega^2 | \star dh + B|^2 + | \star dB + V(x)h|^2 \,.$$

Proof. The above inequality follows immediately from applying the conclusion of Proposition 7.1.1 to the estimate (7.5) and (7.6). □

7.2 Combining near and far estimates

In this section we combine local weighted estimates near the vortex set and uniform elliptic estimates far from the vortex set with a concentration compactness type argument. Roughly speaking, we show that the *discrepancy* cannot concentrate in an intermediate annulus around the vortex set. This allows us to *glue* the near and far estimates:

Proof of Theorem 7.0.1. Using the first assumption we can gauge fix (u, ∇) in such a way that u and u_0 have the same phase almost everywhere; precisely:

$$(u, \nabla) \longrightarrow (u\gamma, \nabla - i\gamma^*(d\theta)) \text{ where } \gamma = \frac{u_0}{|u_0|}(\frac{u}{|u|})^{-1} \,.$$

Note that $d(\gamma^*(d\theta)) = d((\frac{u_0}{|u_0|})^*(d\theta)) - d((\frac{u}{|u|})^*(d\theta)) = 0$ since u and u_0 have the same zero set, counted with multiplicity. Then we can write the discrepancy as:

$$\begin{aligned} E(re^{i\theta}, A) - 2\pi N &= \int_{\mathbb{R}^2} r^2 | \star d\log(r) + A - d\theta|^2 + | \star dA - \frac{1-r^2}{2}|^2 \\ &= \int_{\mathbb{R}^2} r^2 | \star dh + B|^2 + | \star dB + r_0^2 \frac{e^{2h}-1}{2}|^2 \,, \end{aligned}$$

where $h = \log\left(\frac{r}{r_0}\right)$ and $B = A - A_0$. By the third assumption we have the bound $\|h\|_{L^\infty(\mathbb{R}^2)} \leq$

$\log(\Lambda)$ which also means $\frac{e^{2h}-1}{2}$ is comparable with h:

$$\left(\frac{1-\Lambda^{-2}}{2\log\Lambda}\right) \leq \left(\frac{e^{2h}-1}{2h}\right) \leq \left(\frac{\Lambda^{2}-1}{2\log\Lambda}\right),$$

so we can write:

$$E(re^{i\theta}, A) - 2\pi N \geq \int_{\mathbb{R}^2} \Lambda^{-2} r_0^2 |\star dh + B|^2 + |\star dB + V(x)h|^2 ,$$

for some positive potential $\left(\frac{1-\Lambda^{-2}}{2\log\Lambda}\right) r_0^2(x) \leq V(x) \leq \left(\frac{\Lambda^{2}-1}{2\log\Lambda}\right) r_0^2(x)$. Now by Proposition 6.2.1 we know that r_0 behaves like the weights defined in Proposition 7.1.1; since h is not necessarily compactly supported, in order to apply the estimates Proposition 7.1.1, we use a concentration compactness type argument. Our goal is to prove that there exists a constant $C_{\Lambda,N} > 0$ such that:

$$\int_{\mathbb{R}^2} r_0^2 |h|^2 \leq C_{\Lambda,N} \int_{\mathbb{R}^2} r_0^2 |\star dh + B|^2 + |\star dB + V(x)h|^2 . \tag{7.1}$$

Arguing by contradiction, there exists a sequence $\{r_{0^k}, h_k, B_k, V_k\}_{k=1}^{\infty}$ such that:

$$\int_{\mathbb{R}^2} r_{0^k}^2 |h_k|^2 = 1 ,$$

$$\eta_k^2 = \int_{\mathbb{R}^2} r_{0^k}^2 |\star dh_k + B_k|^2 + |\star dB_k + V_k(x)h_k|^2 \longrightarrow 0 \text{ as } k \longrightarrow \infty .$$

Now by Proposition 6.2.1 (applied for each k) there exists M_k balls $B_{\rho_{k,1}}(z_{k,1}), \ldots, B_{\rho_{k,M_k}}(z_{M_k})$ for some $M_k \leq N$ such that $C_N^{-1} \leq \rho_{k,j} \leq C_N$ and $\{r_{0^k} \leq \beta_k\} \subset \cup_{j=1}^{M_k} B_{\rho_{k,j}}(z_{k,j})$ for some $\frac{1}{4} < \beta_k < \frac{1}{2}$ and $B_{2\rho_{k,i}}(z_{k,i}) \cap B_{2\rho_{k,j}}(z_{k,j}) = \emptyset$ for all $1 \leq i < j \leq M_k$. Now take ϕ_k, ψ_k and χ_k to be three smooth cut-off functions on $\mathbb{R}^2$ defined as follows:

$$
\begin{cases}
\phi_k = 0 & \text{on } \mathbb{R}^2 \backslash \bigcup_{j=1}^{M_k} B_{2\rho_{k,j}}(z_{k,j})\,, \\
\phi_k = 1 & \text{on } \bigcup_{j=1}^{M_k} B_{(1+\frac{2}{3})\rho_{k,j}}(z_{k,j})\,,
\end{cases}
$$

$$
\begin{cases}
\psi_k = 0 & \text{on } \bigcup_{j=1}^{M_k} B_{(1+\frac{1}{3})\rho_{k,j}}(z_{k,j})\,, \\
\psi_k = 1 & \text{on } \mathbb{R}^2 \backslash \bigcup_{j=1}^{M_k} B_{(1+\frac{2}{3})\rho_{k,j}}(z_{k,j})\,,
\end{cases}
$$

$$
\begin{cases}
\chi_k = 0 & \text{on } \mathbb{R}^2 \backslash \bigcup_{j=1}^{M_k} \left(B_{2\rho_{k,j}}(z_{k,j}) \backslash B_{\rho_{k,j}}(z_{k,j})\right), \\
\chi_k = 1 & \text{on } \bigcup_{j=1}^{M_k} \left(B_{(1+\frac{2}{3})\rho_{k,j}}(z_{k,j}) \backslash B_{(1+\frac{1}{3})\rho_{k,j}}(z_{k,j})\right),
\end{cases}
$$

with the point-wise estimate $|d\phi_k| + |d\psi_k| + |d\chi_k| \leq C_N$. Note that this is possible since $C_N^{-1} \leq \rho_{k,j} \leq C_N$. Then there exists $e_1 \in L^2(\mathbb{R}^2;\mathbb{R}^2)$ and $e_2 \in L^2(\mathbb{R}^2;\mathbb{R})$ such that:

$$
-\Delta h_k + V_k(x) h_k = - \div (\frac{e_1}{r_{0^k}}) + e_2\,, \tag{7.2}
$$

$$
\|e_1\|^2_{L^2(\mathbb{R}^2)} + \|e_2\|^2_{L^2(\mathbb{R}^2)} = \eta_k^2\,.
$$

We test (7.2) with $\psi_k h_k$, apply Young's inequality and use the estimates on r_{0^k} in Proposition 6.2.1 inside the annulus $B_{2\rho_{k,j}} \backslash B_{\rho_{k,j}}(z_{k,j})$ to see that

$$
\int_{\mathcal{B}_k} |h_k|^2 + |dh_k|^2 \leq C_N(\eta_k^2 + \int_{\mathcal{A}_k} |h_k|^2)\,, \tag{7.3}
$$

$$
\text{where } \mathcal{B}_k = \mathbb{R}^2 \backslash \bigcup_{j=1}^{M_k} B_{(1+\frac{2}{3})\rho_{k,j}}(z_{k,j}) \;\text{ and }\; \mathcal{A}_k = \bigcup_{j=1}^{M_k} \left(B_{(1+\frac{2}{3})\rho_{k,j}} \backslash B_{(1+\frac{1}{3})\rho_{k,j}}(z_{k,j})\right).
$$

Now we apply Proposition 7.1.1 and Proposition 6.2.1 to $\phi_k h_k$, $\phi_k B_k$, together with $C_\Lambda^{-1} r_{0^k}^2 \leq V_k \leq$

$C_\Lambda r^2_{0^k}$; then we collect the boundary terms using the bound $|d\phi_k| \le C_N$ to see that:

$$\int_{\mathcal{B}_k^c} r^2_{0^k}|h_k|^2 \le C_{\Lambda,N}\int_{\mathbb{R}^2} r^2_{0^k}|\star d(\phi_k h_k) + (\phi_k B_k)|^2 + |\star d(\phi_k B_k) + V_k(x)(\phi_k h_k)|^2$$
$$\le C_{\Lambda,N}(\eta_k^2 + \int_{\mathcal{B}_k}|h_k|^2 + |dh_k|^2) \le_{(7.3)} C_{\Lambda,N}(\eta_k^2 + \int_{\mathcal{A}_k}|h_k|^2)\,.$$

Then we add the estimate above to (7.3) to see the lower bound:

$$1 = \int_{\mathbb{R}^2} r^2_{0,k}|h_k|^2 \le C_{\Lambda,N}(\eta_k^2 + \int_{\mathcal{A}_k}|h_k|^2)\,.$$

So we get that for large enough k, for a possibly different constant:

$$\int_{\mathcal{A}_k}|h_k|^2 \ge C_{\Lambda,N} > 0\,.$$

Finally testing (7.2) with $\chi_k h_k$ we get that:

$$\int_{\mathcal{A}_k}|h_k|^2 + |dh_k|^2 \le C_{\Lambda,N}\,. \tag{7.4}$$

Notice that by definition $\mathcal{A}_k = \bigcup_{j=1}^{M_k}\left(B_{(1+\frac{2}{3})\rho_{k,j}}\backslash B_{(1+\frac{1}{3})\rho_{k,j}}(z_{k,j})\right)$ is a disjoint union of $M_k \le N$ annuli so we get that there is at least one annulus in which the energy is concentrating:

$$\int_{\mathcal{A}_{k,0}}|h_k|^2 \ge C_{\Lambda,N} \text{ where } \hat{\mathcal{A}}_k = B_{(1+\frac{2}{3})\rho_{k,j_0}}\backslash B_{(1+\frac{1}{3})\rho_{k,j_0}}(z_{k,j_0})\,. \tag{7.5}$$

Then we use Corollary 7.1.2, Theorem 3.1.1 and similar computations in the preceding paragraphs to get uniform bounds as follows:

$$\|r_{0^k}^{1+\frac{1}{2N}}h_k\|^2_{W^{1,2}(\mathbb{R}^2)} \le C_{\Lambda,N}\,. \tag{7.6}$$

Note that if the right hand side of (7.1) is zero we get that $-\Delta h + V(x)h = 0$. Testing with h and integrating by parts we get that $h = 0$. From the uniform bounds (7.4) and (7.6) we get that there exists a sub-sequence (not relabeled) such that:

$$\tilde{r}_{0^k}^{1+\frac{1}{2N}} \tilde{h}_k \rightharpoonup 0 \text{ weakly in } H^1(\mathbb{R}^2)\,,$$
$$\tilde{h}_k \longrightarrow 0 \text{ strongly in } L^2(B_{1+\frac{2}{3}} \backslash B_{1+\frac{1}{3}})\,,$$
$$\text{where } \tilde{r}_{0^k}^{1+\frac{1}{2N}} \tilde{h}_k(x) = r_{0^k}^{1+\frac{1}{2N}} h_k(\rho_{k,j_0} x + z_{k,j_0})\,.$$

This contradicts the lower bound (7.5) on the annulus $\hat{\mathcal{A}}_k$ and we conclude. □

Proof of Theorem 5.1.2. We apply Corollary 7.1.2 combined with the estimate (7.3) on each annulus and conclude the proof. □

7.3 Stability of the Jacobian

Recall the definition of the Yang-Mills-Higgs Jacobian:

$$J(u, \nabla) = \Psi(u) + \omega(1 - |u|^2) \text{ where } \Psi(u)(j,k) = 2\langle \nabla_{\partial_j} u, i\nabla_{\partial_k} u\rangle\,,$$

for $1 \leqslant j, k \leqslant 2$. Here ω is the real curvature two-form associated to F_∇. Note that the definition of the Yang-Mills-Higgs Jacobian is gauge invariant. We recall the statement of Theorem 5.1.3:

Theorem. *For any $N \in \mathbb{N}$ and $\Lambda > 1$ there exists $C_{\Lambda,N}, \eta_{\Lambda,N} > 0$ with the following properties. Let $(u, \nabla) \in W^{1,2}_{loc}(\mathbb{R}^2)$ be a section and connection on the trivial line bundle $\mathbb{R}^2 \times \mathbb{C}$ such that*

(i) $\star d\left((\frac{u}{|u|})^*(d\theta)\right) = 2\pi \sum_{k=1}^N \delta_{x_k}$ *for a collection of points $\{x_k\}_{k=1}^N \subset \mathbb{R}^2$ (counted with multiplicity).*

(ii) $E(u, \nabla) - 2\pi N \leq \eta^2_{\Lambda,N}$.

(iii) $\Lambda^{-1}|u_0| \leq |u| \leq \Lambda|u_0|$ *for some* N*-vortex solution* (u_0, ∇_0) *with* $\{x_k\}_{k=1}^N$ *as the zero set (counted with multiplicity).*

Then we have the following estimate:

$$\int_{\mathbb{R}^2} |J(u,\nabla) - J(u_0,\nabla_0)| \leq C_{\Lambda,|N|}\sqrt{E(u,\nabla) - 2\pi|N|}\,.$$

Proof. Recall the estimate of Theorem 5.1.2 for any $0 < \varepsilon < \frac{1}{N}$

$$\int_{\mathbb{R}^2} |u_0|^{2+2\varepsilon}\left[|A - A_0|^2 + \left|d\log\left(\frac{|u|}{|u_0|}\right)\right|^2\right] \leq \frac{C_{\Lambda,|N|}}{\varepsilon^2}\left[E(u,\nabla) - 2\pi|N|\right]\,,$$

Note that we can write the Jacobian in a gauge invariant way. Using *(i)* we can gauge fix as in the proof of Theorem 7.0.1 such that $u = re^{i\theta}$ and $u_0 = r_0 e^{i\theta}$ have the same phase. Then we can rewrite the Yang-Mills-Higgs Jacobian as follows:

$$J(u,\nabla) = (1 - r^2)dA - 2rdr \wedge (A - d\theta)\,.$$

To estimate the difference, we see that:

$$\begin{aligned}\int_{\mathbb{R}^2} |J(u,\nabla) - J(u_0,\nabla_0)| &\leq \int_{\mathbb{R}^2} \left|(1-r^2)dA - (1-r_0^2)dA_0\right| \\ &\quad + \int_{\mathbb{R}^2} |2rdr\wedge(A - d\theta) - 2r_0dr_0\wedge(A_0 - d\theta)| = \mathbf{I} + \mathbf{II}\,.\end{aligned}$$

We use Theorem 7.0.1 and estimate the first term:

$$\begin{aligned}\mathbf{I} &\leq \int_{\mathbb{R}^2} \left|(1-r^2)dA - (1-r_0^2)dA_0\right| \\ &\leq \left[\int_{\mathbb{R}^2} |r^2 - r_0^2|^2\right]^{\frac{1}{2}}\left[\int_{\mathbb{R}^2} |dA|^2\right]^{\frac{1}{2}} + \left[\int_{\mathbb{R}^2} (1-r_0^2)^2\right]^{\frac{1}{2}}\left[\int_{\mathbb{R}^2} |dA - dA_0|^2\right]^{\frac{1}{2}} \\ &\leq C_{\Lambda,|N|}\sqrt{E(u,\nabla) - 2\pi|N|}\,.\end{aligned}$$

Then for the second term we proceed as follows:

$$
\begin{aligned}
\mathbf{II} \leq & \int_{\mathbb{R}^2} |rdr \wedge (A - d\theta) - r_0 dr_0 \wedge (A_0 - d\theta)| \\
\leq & C_\Lambda \left[\int_{\mathbb{R}^2} r_0^{2\varepsilon-2} |rdr - r_0 dr_0|^2 \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^2} r_0^{2-2\varepsilon} |A_0 - d\theta|^2 \right]^{\frac{1}{2}} \\
+ & C_\Lambda \left[\int_{\mathbb{R}^2} r_0^{2+2\varepsilon} |A - A_0|^2 \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^2} r_0^{-2\varepsilon} |dr_0|^2 \right]^{\frac{1}{2}} .
\end{aligned}
\tag{7.1}
$$

Take $\frac{1}{3N} < \varepsilon < \frac{1}{N}$ and estimate as follows:

$$
\begin{aligned}
& \int_{\mathbb{R}^2} r_0^{2\varepsilon-2} |rdr - r_0 dr_0|^2 = \\
& \quad = \int_{\mathbb{R}^2} r_0^{2\varepsilon-2} \left| r^2 d\log(r) - r_0^2 d\log(r_0) \right|^2 \\
& \quad \leq C_\Lambda \int_{\mathbb{R}^2} r_0^{2+2\varepsilon} \left| d\log\left(\frac{r}{r_0}\right) \right|^2 + r_0^{2\varepsilon-2} |d\log(r_0)|^2 (r^2 - r_0^2)^2 \\
& \quad \leq C_{\Lambda,|N|} \left[E(u,\nabla) - 2\pi|N| \right] + C_\Lambda \int_{\mathbb{R}^2} r_0^{2\varepsilon+2} |d\log(r_0)|^2 \left| \log\left(\frac{r}{r_0}\right) \right|^2 .
\end{aligned}
\tag{7.2}
$$

In the last line we used assumption *(iii)* to see that

$$
C_\Lambda^{-1} \left| \log\left(\frac{r}{r_0}\right) \right| \leq \left| \frac{r}{r_0} - 1 \right| \leq C_\Lambda \left| \log\left(\frac{r}{r_0}\right) \right| .
$$

Now let $\{B_{\rho_k}(z_k)\}_{k=1}^{M}$ be the covering constructed in Proposition 6.2.1 and let ω_k the associated weights. Then locally we have:

$$
\| d\log(r_0) - d\log(\omega_k) \|_{L^\infty(B_{2\rho_k}(z_k))} \leq C_N .
$$

Moreover take smooth indicators ϕ_k for $B_{\rho_k}(z_k)$ such that $\phi = 1$ on $B_{\rho_k}(z_k)$ and zero outside

$B_{2\rho_k}(z_k)$ with $|d\phi_k| \le C_N$; we see that the last term in (7.2) is bounded as follows:

$$\int_{\mathbb{R}^2} r_0^{2\varepsilon+2}|d\log(r_0)|^2 \left|\log\left(\frac{r}{r_0}\right)\right|^2 \le C_{N,\Lambda}\sum_{k=1}^{M}\int_{B_{2\rho_k}(z_k)} \phi_k^2\omega_k^{2\varepsilon+2}(1+|d\log(\omega_k)|^2)\left|\log\left(\frac{r}{r_0}\right)\right|^2$$
$$+C_{N,\Lambda}\int_{\mathbb{R}^2\setminus\bigcup_{k=1}^{M}B_{\rho_k}(z_k)} |\log\left(\frac{r}{r_0}\right)|^2$$

Then we use Theorem 3.1.1 with $\omega_k^{2+2\varepsilon}$ and Poincaré inequality for the first sum in the above display to see that:

$$\begin{aligned}
&\sum_{k=1}^{M}\int_{B_{2\rho_k}(z_k)} \phi_k^2\omega_k^{2\varepsilon+2}(C+|d\log(\omega_k)|^2)\left|\log\left(\frac{r}{r_0}\right)\right|^2 \\
&\le C_{N,\Lambda}\sum_{k=1}^{M}\int_{B_{2\rho_k}(z_k)} \left|\phi_k\omega_k^{1+\varepsilon}\log\left(\frac{r}{r_0}\right)\right|^2 + \phi_k^2|d(\omega_k^{1+\varepsilon})|^2\left|\log\left(\frac{r}{r_0}\right)\right|^2 \\
&\le C_{N,\Lambda}\sum_{k=1}^{M}\int_{B_{2\rho_k}(z_k)} \phi_k^2\omega_k^{2+2\varepsilon}\left|d\log\left(\frac{r}{r_0}\right)\right|^2 \le C_{N,\Lambda}\left[E(u,\nabla)-2\pi|N|\right] .
\end{aligned}$$

Here the last line follows by Corollary 7.1.2 (or Theorem 5.1.2). We then use (7.3) to bound $\log(r/r_0)$ on $\mathbb{R}^2\setminus\bigcup_{k=1}^{M}B_{\rho_k}(z_k)$ and see that:

$$\int_{\mathbb{R}^2} r_0^{2\varepsilon+2}|d\log(r_0)|^2\left|\log\left(\frac{r}{r_0}\right)\right|^2 \le C_{N,\Lambda}\left[E(u,\nabla)-2\pi|N|\right] .$$

Using the above display we can bound (7.2) as follows:

$$\int_{\mathbb{R}^2} r_0^{2\varepsilon-2}|rdr-r_0dr_0|^2 \le C_{\Lambda,N}\left[E(u,\nabla)-2\pi|N|\right] .$$

Now all that remains is to bound the second term of (7.2). Notice that if $\varepsilon < \frac{1}{N}$ the term $r_0^{-2\varepsilon}$ is comparable to $|x-p|^{2-\delta}$ around any vortex point p for some $\delta > 0$. This means that it is locally integrable, hence

$$\int_{\mathbb{R}^2} r_0^{-2\varepsilon} |dr_0|^2 < C_{N,\varepsilon} \,.$$

Here we used that the integral of $|dr_0|^2$ is bounded on the whole domain. Now we use the estimates on the connection in Theorem 5.1.2 to see that for $\varepsilon = \frac{1}{2N}$:

$$\mathbf{II} \leq C_{\Lambda,N} \sqrt{E(u, \nabla) - 2\pi|N|} \,.$$

This is indeed the desired conclusion. □

Chapter 8

A selection principle and the proof of stability

The main goal of this section is to drop the first and the third assumption in Theorem 7.0.1. We prove that for any $(u, \nabla) \in W^{1,2}_{loc}(\mathbb{R}^2)$ that nearly minimizes the energy, we can *select* another pair $(\tilde{u}, \tilde{\nabla})$ close enough to (u, ∇) that satisfies the assumptions of Theorem 7.0.1. We do this by inductively replacing (u, ∇) with a minima of the penalized functional (8.1) (proof of existence in Lemma 8.1.1). In each iteration while staying close to the original (u, ∇) we gain at least one derivative of regularity (proof of regularity in Lemma 8.2.1). Then using Lemma A.0.1 (smooth perturbation of complex polynomials) we show that after finitely many steps the new pair $(\tilde{u}, \tilde{\nabla})$ satisfies the assumptions of Theorem 7.0.1.

Theorem 8.0.1. *There exists constants $C_N, \Lambda_N > 0$ with the property that for any N-vortex section and connection (u, ∇) with finite energy $E(u, \nabla) = 2\pi|N| + \eta^2$ there exists another N-vortex section and connection $(\tilde{u}, \tilde{\nabla})$ such that:*

(i) $\|u - \tilde{u}\|^2_{L^2(\mathbb{R}^2)} + \|F_\nabla - F_{\tilde{\nabla}}\|^2_{L^2(\mathbb{R}^2)} \leq C_N \eta^2$,

(ii) $\Lambda_N^{-1}|u_0| \leq |\tilde{u}| \leq \Lambda_N |u_0|$ *for some N-vortex solution (u_0, ∇_0) to the vortex equations (6.4),*

(iii) $2\pi N \leq E(\tilde{u}, \tilde{\nabla}) \leq E(u, \nabla)$,

provided that η is small enough.

To prove Theorem 8.0.1 we use a penalized energy to find a *selection principle*:

$$\mathcal{G}_{(u,\nabla)}(u_1, \nabla_1) = E(u_1, \nabla_1) + \|u_1 - u\|^2_{L^2(\mathbb{R}^2)} + \|A_1 - A\|^2_{L^2(\mathbb{R}^2)} \,. \tag{8.1}$$

This energy also enjoys the coupled gauge invariance

$$(u, \nabla) \to (ue^{i\xi}, A + d\xi) \text{ and } (u_1, \nabla_1) \to (u_1 e^{i\xi}, A_1 + d\xi)\,,$$

for any smooth compactly supported function $\xi \in C_c^\infty(\mathbb{R}^2)$. Note that both pairs have to be gauge transformed with the same function.

8.1 Existence

We first prove the existence of a minimizer of (8.1) via the direct method of the calculus of variations:

Lemma 8.1.1. *For any N-vortex section and connection (u, ∇) with $E(u, \nabla) = 2\pi N + \eta^2$, the penalized energy $\mathcal{G}_{(u,\nabla)}$ (8.1) achieves its minimum for some N-vortex pair $(u_1, \nabla_1) \in W^{1,2}_{loc}(\mathbb{R}^2)$.*

Proof. We have the lower bound:

$$\mathcal{G}_{(u,\nabla)}(u_1, \nabla_1) \geq E(u_1, \nabla_1) \geq 2\pi N \,.$$

Hence there exists an N-vortex sequence (v_j, ∇_j) realizing the infimum of (8.1):

$$\lim_{j\to\infty} \mathcal{G}_{(u,\nabla)}(v_j, \nabla_j) = \inf_{(u_1,\nabla_1)\in W^{1,2}_{loc}} \mathcal{G}_{(u,\nabla)}(u_1, \nabla_1) \,.$$

Let Ω be any bounded smooth simply connected domain. By the gauge invariance of (8.1), we can gauge fix $(v_j, \nabla_j) \to (w_j, B_j)$ in the Coulomb gauge such that B_j is divergence free $d^* B_j = 0$ inside the domain Ω and its normal component vanishes $\iota_\nu B_j = 0$ on the boundary $\partial\Omega$. By Gaffney inequalities (see [50, Theorem 4.8]) and a compactness argument we can bound:

$$\|B_j\|^2_{W^{1,2}(\Omega)} \leq C_\Omega \|dB_j\|^2_{L^2(\Omega)} \leq C_{\Omega,N} \,.$$

By the global Sobolev embedding $W^{1,2}(\mathbb{R}^2) \hookrightarrow_c L^4(\mathbb{R}^2)$ we get the following bounds:

$$\begin{aligned}
\|w_j\|^2_{W^{1,2}(\Omega)} &= \int_\Omega |w_j|^2 + |dw_j|^2 \\
&\leq C_\Omega \left(1 + \sqrt{\int_\Omega (1 - |w_j|^2)^2}\right) + C \int_\Omega |dw_j - i w_j B_j|^2 + |w_j B_j|^2 \\
&\leq C_{\Omega,N} + C\|w_j B_j\|^2_{L^2(\Omega)} \leq C_{\Omega,N} + C\|w_j\|^2_{L^4(\Omega)} \|B_j\|^2_{L^4(\Omega)} \\
&\leq C_{\Omega,N} + C\|w_j\|^2_{W^{1,2}(\Omega)} \|B_j\|^2_{W^{1,2}(\Omega)} \leq C_{\Omega,N} \,.
\end{aligned}$$

Then we can find a sub-sequence (w_j, B_j) (not necessarily relabeled) and a limit section and connection $(u_1, \nabla_1) \in W^{1,2}_{loc}$ such that after gauge fixing (u_1, A_1) in the Coulomb gauge in Ω we get that:

$$(w_j, B_j) \rightharpoonup (u_1, A_1) \text{ weakly in } W^{1,2}_{loc}(\mathbb{R}^2) \,.$$

We need to show that (u_1, ∇_1) is also an N-vortex pair. First observe that (u, ∇) is a competitor with energy $\mathcal{G}_{(u,\nabla)}(u, \nabla) = 2\pi N + \eta^2$; by lower semi-continuity we get that:

$$\|u_1 - u\|^2_{L^2(\mathbb{R}^2)} \leq \liminf_{j\to\infty} \|v_j - u\|^2_{L^2(\mathbb{R}^2)} \leq \eta^2 \,.$$

In particular the difference is bounded in L^2. Now consider a smooth kernel ϕ and let the ε-

rescaled version to be $\phi_\varepsilon = \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon})$. Then consider the mollified functions $u * \phi_\varepsilon$ and $u_1 * \phi_\varepsilon$. By the embedding $W^{1,2}_{loc}(\mathbb{R}^2) \to \mathrm{VMO}_{loc}(\mathbb{R}^2)$, where VMO_{loc} is the space of functions with locally vanishing mean oscillation, we can see that:

$$u * \phi_\varepsilon \to u \text{ and } u_1 * \phi_\varepsilon \to u_1 \text{ in } \mathrm{BMO} \cap L^1_{loc}(B_R) \text{ as } \varepsilon \to 0\,.$$

By [14, Property 2 in Chapter II.2] we can see that for small enough ε the degree of $u * \phi_\varepsilon$ and $u_1 * \phi_\varepsilon$ is equal to the degree of u and u_1, respectively. Now we can estimate for fixed ε:

$$\lim_{R\to\infty} \|\phi_\varepsilon * (u - u_1)\|_{C^0(\mathbb{R}^2\setminus B_R)} \le \lim_{R\to\infty} \varepsilon^{-1}\|u - u_1\|_{L^2(\mathbb{R}^2\setminus B_R)} = 0\,.$$

Hence for all R large enough, the degree of $\phi_\varepsilon * u$ coincides with $\phi_\varepsilon * u_1$ and we can conclude that (u_1, ∇_1) is indeed an N-vortex pair. □

Remark 8.1.2. Without loss of generality we may assume $|u| \le 3$, precisely: For any (u, ∇) with $\eta > 0$ small enough, we can find another section u_1 such that $|u_1| \le 3$ and:

$$\|u - u_1\|^2_{L^2(\mathbb{R}^2)} \le C\eta^2 \text{ and } E(u_1, \nabla) \le E(u, \nabla)\,,$$

for some universal constant C.

Proof. Identifying $u = re^{i\theta}$ and $\nabla : d - iA$, consider the super level set $\{r \ge 2\}$. From the energy bound we can see that:

$$|\{r \ge 2\}| + \int_{\{r\ge 2\}} |dr|^2 \le C_N\,.$$

Similar to the proof of Lemma 6.1.1, using the coarea formula and the mean value theorem we can find $2 < \beta < 3$ such that $\mathcal{H}^1\left(\partial\{r \ge \beta\}\right) \le C_N$. Since perimeter bounds diameter, $\{r \ge \beta\}$ is made of a collection of bounded simply connected domains. Now we unwrap the discrepancy in this domain (with the Bogomolny trick, for more details see Lemma 6.1.1):

$$\eta^2 \geq \int_{\{r\geq\beta\}} |\star dr + r(A - d\theta)|^2 + |\star dA - \frac{1-r^2}{2}|^2$$
$$= \int_{\{r\geq\beta\}} |dr|^2 + r^2|A - d\theta|^2 + |dA|^2 + \frac{(1-r^2)^2}{4} - d(d\theta)(1-r^2)\,.$$

Note that $d(d\theta) = 0$ away from $\{r = 0\}$, hence we can see that:

$$\int_{\{r\geq\beta\}} \frac{(1-r^2)^2}{4} \leq \eta^2\,. \tag{8.1}$$

Since $(1-r^2)^2 > 9$ on $\{r \geq \beta\}$ hence (8.1) tells us that:

$$|\{r \geq \beta\}| \leq \eta^2\,.$$

Now we define u_1:

$$u_1 = \begin{cases} u & \text{on } \{|u| < 3\}\,, \\ 3\frac{u}{|u|} & \text{on } \{|u| \geq 3\}\,. \end{cases}$$

We estimate the energy difference:

$$E(u,\nabla) - E(u_1,\nabla) = \int_{\{r\geq 3\}} |dr|^2 + (r^2-9)|A - d\theta|^2 + |dA|^2 + \frac{(1-r^2)^2 - 64}{4} \geq 0\,,$$

and the difference of sections using (8.1):

$$\|u - u_1\|^2_{L^2(\mathbb{R}^2)} = \int_{\{r\geq 3\}} |u_1 - u|^2 \leq C|\{r \geq 3\}| + C\int_{\{r\geq 3\}} r^2 \leq C\eta^2\,,$$

where the last inequality followed from

$$\int_{\{r\geq 3\}} r^2 \leq C\int_{\{r\geq 3\}} \frac{(1-r^2)^2}{4} \leq C\eta^2\,.$$

This indeed gives us the desired conclusion. □

8.2 Regularity

In this section we derive regularity estimates for minimizers of the penalized functional (8.1):

Lemma 8.2.1. *Let (u,∇) be a section and connection with finite energy $E(u,\nabla) = 2\pi N + \eta^2$, with small enough $\eta > 0$ and $|u| \leq 3$. Moreover let (u_1,∇_1) be a minimizer of (8.1). Then for any $0<\alpha<1$ we have the regularity estimates below:*

(i) $\|u_1\|_{C^{1,\alpha}(\Omega)} + \|A_1\|_{C^{1,\alpha}(\Omega)} \leq C_{\Omega,N,\alpha}$,

(ii) $\|u_1\|_{C^{k+1,\alpha}(\Omega)} + \|A_1\|_{C^{k+1,\alpha}(\Omega)} \leq C_{\Omega,N,k,\alpha}(\|u\|_{C^{k,\alpha}(\Omega)} + \|A\|_{C^{k,\alpha}(\Omega)} + 1)$,

for all $k \geq 1$ and any domain Ω where both (u_1,A_1), (u,A) are independently measured in the Coulomb gauge in a slightly bigger domain $U \supset \Omega$.

Proof. We inspire from the strategy in [62, Appendix]. However since we are in two dimensions, the embedding $W^{2,2}(\mathbb{R}^2) \hookrightarrow C^\alpha(\mathbb{R}^2)$ greatly simplifies the proof.

For any open bounded smooth domain Ω we take three bigger domains $\Omega \subset \Omega_1 \subset \Omega_2 \subset \Omega_3$. Then we gauge fix (u_1,A_1) in Ω_3 in the Coulomb gauge such that $d^*A_1 = 0$ inside Ω_3 and $A_1(\nu) = 0$ on ∂U. Now the Euler Lagrange equations for minimizers of (8.1) are as follows:

$$\Delta u_1 = 2\langle idu_1, A_1\rangle + |A_1|^2 u_1 - \frac{1}{2}(1-|u_1|^2)u_1 + (u_1 - u)\,, \tag{8.1}$$

$$\Delta_H A_1 = \langle du_1 - iu_1A_1, iu_1\rangle + A_1 - A\,. \tag{8.2}$$

Here Δ_H is the Laplace Beltrami operator for one-forms. The Euler Lagrange equations for the difference in gauge is as follows:

$$d^*A = \langle u, iu_1 \rangle \,. \tag{8.3}$$

By Gaffney type inequalities in [50] we estimate:

$$\|A_1\|_{W^{1,2}(\Omega_3)} \leq C\|dA_1\|_{L^2(\Omega_3)} \leq C_N \,.$$

The Euler-Lagrange equation for $|u_1|^2$ is as follows:

$$\Delta\frac{1}{2}|u_1|^2 = |\nabla_1 u_1|^2 - \frac{1}{2}(1-|u_1|^2)|u_1|^2 + |u_1|^2 - \langle u, u_1 \rangle \,. \tag{8.4}$$

We apply the maximum principle for (8.4) and the bound $|u| \leq 3$ to deduce that $|u_1| \leq \max(|u|, 1) \leq 3$. Then by (8.2) we get that:

$$\|A_1\|_{W^{2,2}(\Omega_2)} \leq C_{N,\Omega_2,\Omega_3} \,.$$

Now by the Sobolev embedding $W^{2,2}(\mathbb{R}^2) \subset C^\alpha(\mathbb{R}^2)$ for any $0 < \alpha < 1$ and $W^{2,2}(\mathbb{R}^2) \subset W^{1,p}(\mathbb{R}^2)$ for all $1 \leq p < \infty$ we estimate that:

$$\|A_1\|_{C^\alpha(\Omega_2)} \leq C_{\alpha,N,\Omega_2,\Omega_3} \text{ and } \|A_1\|_{W^{1,p}(\Omega_2)} \leq C_{p,N,\Omega_2,\Omega_3} \,.$$

Then we use this pointwise bound with (8.1) to see that Δu_1 is bounded in L^2. By standard elliptic estimates we get that:

$$\|u_1\|_{W^{2,2}(\Omega_1)} \leq C_{N,\Omega_1,\Omega_2,\Omega_3} \Rightarrow \|u_1\|_{C^\alpha(\Omega_1)} \leq C_{\alpha,\Omega_1,\Omega_2,\Omega_3} \,.$$

We use the embedding $W^{2,2}(\mathbb{R}^2) \subset W^{1,p}(\mathbb{R}^2)$ again to see that:

$$\|u_1\|_{W^{1,p}(\Omega_1)} \le C_{N,\Omega_1,\Omega_2,\Omega_3}\,.$$

Then by (8.1) and (8.2) we get that Δu_1 and $\Delta_H A_1$ are both bounded in L^p for all $1 < p < \infty$, so we can improve the $W^{2,2}$ estimates to $W^{2,p}$ as follows:

$$\|u_1\|_{W^{2,p}(\Omega)} + \|A_1\|_{W^{2,p}(\Omega)} \le C_{N,\Omega,p}\,.$$

From the embedding $W^{2,p}(\mathbb{R}^2) \subset C^{1,\alpha}(\mathbb{R}^2)$ for $p > 2$ and $\alpha = 1 - \frac{2}{p}$ we get Holder estimates as follows:

$$\|u_1\|_{C^{1,\alpha}(\Omega)} + \|A_1\|_{C^{1,\alpha}(\Omega)} \le C_{\alpha,N,\Omega}\,. \tag{8.5}$$

To gain higher regularity estimates consider the Hodge decomposition of A in U:

$$A = d\phi + A'\,,$$

Where (u', A') is in the Coulomb gauge, precisely $d^*A' = 0$ in Ω_2 and $A'(\nu) = 0$ on the boundary $\partial\Omega_2$. Then we rewrite the system of equations (8.1) to (8.3) using this decomposition:

$$\begin{aligned}
\Delta u_1 &= 2\langle i du_1, A_1\rangle + |A_1|^2 u_1 - \frac{1}{2}(1 - |u_1|^2)u_1 + (u_1 - e^{i\phi}u')\,,\\
\Delta_H A_1 &= \langle du_1 - iu_1 A_1, iu_1\rangle + A_1 - A' - d\phi\,,\\
\Delta\phi &= \langle u_1 - e^{i\phi}u', iu_1\rangle\,.
\end{aligned} \tag{8.6}$$

We gain higher regularity estimates with standard iteration arguments in intermediate domains starting from the apriori estaimtes (8.5) and Schauder estimates for the system of equations (8.6).

□

Proof of Theorem 8.0.1. Take any section and connection (u, ∇) with energy $E(u, \nabla) = 2\pi N + \eta^2$ with small enough $\eta > 0$. First using Remark 8.1.2 we may assume $|u| \leq 3$, without loss of generality. Then we replace (u, ∇) with a minimizer of the penalized energy $\mathcal{G}_{(u,\nabla)}$ in (8.1) (existence provided by Lemma 8.1.1). In fact we repeat this process N times. Then from Lemma 8.2.1 we use the regularity estimate *(i)* at the first step and *(ii)* inductively after the second step. We end up with a new N-vortex section and connection $(\tilde{u}, \tilde{\nabla})$ with the following estimates for some fixed $0 < \alpha < 1$:

$$\|u - \tilde{u}\|^2_{L^2(\mathbb{R}^2)} + \left[E(\tilde{u}, \tilde{\nabla}) - 2\pi N\right] \leq \eta^2 ,$$
$$\|\tilde{u}\|_{C^{N,\alpha}(\Omega)} + \left\|\tilde{A}\right\|_{C^{N,\alpha}(\Omega)} \leq C_{\Omega,N,\alpha} , .$$

for any smooth connected open domain Ω where $(\tilde{u}, \tilde{A})$ is measured in the Coulomb gauge in a slightly bigger domain ((u, ∇) is also simultaneously gauge transformed). Now we consider the sublevel set $\Omega_{\frac{1}{2}} = \{|\tilde{u}| \leq \frac{1}{2}\}$ and its disjoint connected components $\cup_{k=1}^{\infty} \Omega^k_{\frac{1}{2}} = \Omega_{\frac{1}{2}}$. We start with the *perturbed* vortex equations for $|\tilde{u}|$:

$$\Delta \log(|\tilde{u}|) + \frac{1 - |\tilde{u}|^2}{2} = \star d\left((\frac{\tilde{u}}{|\tilde{u}|})^*(d\theta)\right) + \div(\frac{e_1}{|\tilde{u}|}) + e_2 ,$$
$$\|e_1\|^2_{L^2(\mathbb{R}^2)} + \|e_2\|^2_{L^2(\mathbb{R}^2)} \leq \eta^2 .$$

We test this equation with $(\frac{1}{4} - |\tilde{u}|^2)^+$ and estimate for each component $\Omega^k_{\frac{1}{2}}$:

$$|\Omega^k_{\frac{1}{2}} \cap \{|\tilde{u}| \leq \frac{1}{4}\}| + \int_{\Omega^k_{\frac{1}{2}}} |d|\tilde{u}||^2 \leq C \deg(\tilde{u}, \partial\Omega^k_{\frac{1}{2}}) + C\eta^2 .$$

First notice that for all $k \in \mathbb{N}$ the degree $\deg(\tilde{u}, \Omega^k_{\frac{1}{2}}) \geq 0$ is positive (provided η is small enough). Then consider the set $I_0 \subset \mathbb{N}$ of indices k where $\tilde{u}$ has zero rotation number around $\Omega^k_{\frac{1}{2}}$ and $I \subset \mathbb{N}$ the components with positive rotation number. By mean value theorem there exists a $\frac{1}{8} \leq \beta \leq \frac{1}{4}$

such that:

$$\sum_{k\in I_0} \mathcal{H}^1(\partial\{|\tilde{u}| \le \beta\} \cap \Omega^k_{\frac{1}{2}}) \le C \sum_{k\in I_0} \int_{\frac{1}{8}}^{\frac{1}{4}} \mathcal{H}^1(\partial\{|\tilde{u}| \le t\} \cap \Omega^k_{\frac{1}{2}}) dt\,.$$

Then by the coarea formula and then Young's inequality we get that:

$$\sum_{k\in I_0} \mathcal{H}^1(\partial\{|\tilde{u}| \le \beta\} \cap \Omega^k_{\frac{1}{2}}) \le C \sum_{k\in I_0} \int_{\{|\tilde{u}|\le\beta\}\cap\Omega^k_{\frac{1}{2}}} |d|\tilde{u}|| \le C\eta^2\,,$$

where we also used the measure estimates on $\Omega^k_{\frac{1}{2}}$ for $k \in I_0$. Since for connected sets, perimeter bounds diameter, by a Vitali covering argument we find disjoint balls $B_{\rho_k}(x_k)$ for $k \in I_0$ such that:

$$\sum_{k\in I_0} \rho_k \le C\eta^2 \text{ and } \cup_{k\in I_0} \Omega^k_{\frac{1}{2}} \cap \{|\tilde{u}| \le \beta\} \subset \cup_{k\in I_0} B_{\rho_k}(x_k)\,.$$

Moreover we get that $\frac{1}{8} \le |\tilde{u}| \le 3$ on $\cup_{k\in I_0}\partial B_{\rho_k}(x_k)$. Then by Lipschitz bounds and diameter estimates we can *clear out* the set as follows; note that $\frac{1}{8} \le |\tilde{u}| \le 3$ on the boundary of the balls with zero degree, namely $\cup_{k\in I_0}\partial B_{\rho_k}(x_k)$ and $|d|\tilde{u}|| \le C_N$. However since these balls have small diameter, precisely $\sum_{k\in I_0}\rho_k \le C_N\eta^2$, we see that necessarily $|\tilde{u}| \ge \frac{1}{16}$ inside $\cup_{k\in I_0} B_{\rho_k}(x_k)$ (provided η is small enough). Then for the connected components with positive rotation number we can find balls $B_{\rho_k}(x_k)$ for $k \in I$ with $|I| \le N$ such that:

$$\max_{k\in I} \rho_k \le C_N \text{ and } \cup_{k\in I} \Omega^k_{\frac{1}{2}} \cap \{|\tilde{u}| \le \beta\} \subset \cup_{k\in I} B_{\rho_k}(x_k) \text{ and } |I| \le N\,.$$

Then for each $k \in I$ consider (u_1, A_1) with uniform local $C^{N,\alpha}$ estimates. Arguing by contradiction and Arzela-Ascolli, we deduce that for small enough discrepancy $\eta > 0$ the section u_1 is locally a C^N perturbation of a solution to the vortex equations (6.4):

$$\tilde{u}(z) = u_0(z) + R(z) \text{ for } z \in \cup_{k\in I} B_{\rho_k}(x_k) \text{ such that } \|R\|_{C^N(\cup_{k\in I} B_{\rho_k}(x_k))} \le \varepsilon_\eta\,,$$

where ε_η vanishes as $\eta \to 0$. By Proposition 6.2.1 and [74] we know that solutions are locally, up to a smooth change of gauge, complex polynomials multiplied by an analytic nonzero function. More precisely there exists an analytic function $\Lambda_N^{-1} \le g(z) \le \Lambda_N$ uniformly bounded away from zero and uniform C^N bounds only depending on N such that:

$$\frac{\tilde{u}(z)}{g(z)} = \Pi_{k=1}^{M}(z - a_k) + \frac{R(z)}{g(z)} \text{ for } z \in \cup_{k\in I} B_{\rho_k}(x_k)\,.$$

Finally we are in a position to apply Lemma A.0.1 and conclude that $(\tilde{u}, \tilde{A})$ satisfies the assumptions of Theorem 7.0.1. □

8.3 Proof of stability in $\mathbb{R}^2$

Here we prove the stability in its general form:

Proof of Theorem 5.1.1. By Theorem 8.0.1 we find $(\tilde{u}, \tilde{A})$ satisfying the regularity assumptions of Theorem 7.0.1 for some (u_0, ∇_0). Then we apply Theorem 7.0.1 and estimate:

$$\begin{aligned}
\|u - u_0\|^2_{L^2(\mathbb{R}^2)} &\le 2\|u - \tilde{u}\|^2_{L^2(\mathbb{R}^2)} + 2\|\tilde{u} - u_0\|^2_{L^2(\mathbb{R}^2)} \le C_N\eta^2 \text{ and} \\
\|F_\nabla - F_{\nabla_0}\|^2_{L^2(\mathbb{R}^2)} &\le 2\|F_\nabla - F_{\tilde{\nabla}}\|^2_{L^2(\mathbb{R}^2)} + 2\|F_{\tilde{\nabla}} - F_{\nabla_0}\|^2_{L^2(\mathbb{R}^2)} \le C_N\eta^2,
\end{aligned}$$

and we conclude with the proof. □

8.4 The power 2 is optimal

One may perturb any N-vortex solution (u_0, ∇_0) with a smooth real valued function $h \in C_c^\infty(\mathbb{R}^2)$ to see that:

$$E(u_0 e^h, \nabla_0) - 2\pi N = \int_{\mathbb{R}^2} |u_0|^2 | \star dh|^2 + |u_0|^2 |\frac{e^{2h} - 1}{2}|^2 .$$

We can choose h to be a mollified indicator function of a ball $B_R(x)$ sufficiently far from the vortex set (as in Proposition 6.2.1) to see that:

$$\begin{aligned} E(u_0 e^h, \nabla_0) - 2\pi N &\geq CR^2 \geq C\|h\|^2_{L^2(\mathbb{R}^2)} \\ &\geq C\|u_0 e^h - u_0\|^2_{L^2(\mathbb{R}^2)} \geq C \min_{(u,\nabla)\in\mathcal{F}} \|u_0 e^h - u\|^2_{L^2(\mathbb{R}^2)} . \end{aligned}$$

Taking $R \longrightarrow 0$ we see that there exists a sequence $\{(u_k, \nabla_k)\}_{k=1}^\infty$:

$$E(u_k, \nabla_k) \longrightarrow 2\pi N \text{ and } \lim_{k\to\infty} \frac{E(u_k, \nabla_k) - 2\pi N}{\min_{(u_0,\nabla_0)\in\mathcal{F}} \|u_0 - u_k\|^2_{L^2(\mathbb{R}^2)}} > 0 .$$

Hence we can see that the power 2 may not be improved.

Chapter 9

Stability for compact surfaces

In this section we show that the methods above can be adapted to obtain stability for nontrivial line bundles over compact Riemannian surfaces. The proofs are mostly unchanged with slight modifications. Let (M, g) be a smooth Riemann surface and let $L \longrightarrow M$ be a nontrivial Hermitian line bundle over M. Using a Stokes theorem, we have that for any section and connection (u, ∇):

$$E(u, \nabla) = 2\pi |\deg(L)| + \int_M \left[|\nabla_{\partial_1} u + i\nabla_{\partial_2} u|^2 + |\star \omega - \frac{1 - |u|^2}{2}|^2 \right] . \tag{9.1}$$

Naturally, the *vortex equations* take the same form:

$$\nabla_{\partial_1} u + i\nabla_{\partial_2} u = 0 \text{ and } \star \omega = \frac{1 - |u|^2}{2} . \tag{9.2}$$

Where ω is the real two-form associated to F_∇. Integrating the second equation over M we see that:

$$|\deg(L)| \leq \frac{1}{4\pi} \mathrm{vol}(M) .$$

In [41] García-Prada proves if the condition above is satisfied, once we prescribe the zero set (counted with multiplicity), the solution is unique and smooth. We now recall the statement of Theorem 5.2.1:

Theorem. *Let M be a smooth compact Riemannian surface and $L \longrightarrow M$ a Hermitian line bundle over M with $0 \leq \deg(L) \leq \frac{1}{4\pi}\mathrm{vol}(M)$. Then there exists a constant $C_M > 0$ depending only on M, with the following property: Let $(u, \nabla) \in W^{1,2}(M)$ be a section and connection on L such that $E(u,\nabla) - 2\pi\deg(L)$ is small enough, then:*

$$\min_{(u_0,\nabla_0)\in\mathcal{F}} \|u-u_0\|^2_{L(M)} + \|F_\nabla - F_{\nabla_0}\|^2_{L(M)} \leq C_M\left[E(u,\nabla) - 2\pi\deg(L)\right],$$

where $\mathcal{F}$ is the family all minimizers of the Yang-Mills-Higgs energy on $L \longrightarrow M$.

Remark. Before diving into the proof, note that on compact surfaces the Sobolev space $W^{1,2}(M)$ embeds into the space of functions of vanishing mean oscillation $\mathrm{VMO}(M)$, hence the degree is well defined.

9.1 Preliminary estimates on solutions

Here we prove some facts about solutions of the vortex equations on smooth compact surfaces. First using the vortex equations, we can see that

$$\frac{1}{2}\Delta|u_0|^2 = |\nabla_0 u_0|^2 - \frac{1}{2}|u_0|^2(1-|u_0|^2)$$

and by an application of the maximum principle, since M is compact, we can see that $|u| \leq 1$ on M.

Proposition 9.1.1. *For any compact smooth Riemann surface (M, g) there exists small constants $c_M, \beta_M > 0$ and a constant $C_M > 1$ depending on M with the following property. Let $L \longrightarrow M$ be a*

Hermitian line bundle on M with $0 \leq \deg(L) \leq \frac{1}{4\pi}\mathrm{vol}(M)$ and (u_0, ∇_0) be a solution to the vortex equations (6.4) with the prescribed zero set $x_1, \ldots, x_{\deg(L)} \in M$ counted with multiplicity. Then there exists a geodesic ball $B_\rho(x_0)$ such that:

(i) $|u_0| \geq \beta_M > 0$ *on* $B_{2\rho}(x_0)$ *with* $c_M \leq \rho \leq 2c_M$ *,*

(ii) $C_M^{-1}\omega \leq |u_0| \leq C_M\omega$*, where* $\omega(x) = \Pi_{k=1}^{\deg(L)} d(x, x_k)$ *,*

(iii) All geodesic balls of radius less than $2c_M$ are uniformly bi-Lipschitz to euclidean balls of comparable radius.

Here $d(x, y)$ is the geodesic distance between x, y on M.

Proof. The proof is essentially the same as Proposition 6.2.1 with some modifications in the case of compact surfaces. First notice that the modulus $r_0 = |u_0|$ satisfies a similar equation:

$$-\Delta_g \log(r_0) + \frac{1}{2}(r_0^2 - 1) = -\sum_{k=1}^{\deg(L)} 2\pi\delta_{x_k} \,. \tag{9.1}$$

Multiplying the above display by $(\beta^2 - r_0^2)^+$ and integrating by parts we see that:

$$\int_{\{r_0 \leq \beta\}} |dr_0|^2 + \frac{1}{2}(1 - r_0^2)(\beta^2 - r_0^2) = 2\pi\beta^2 \deg(L) \,.$$

Now we take β_M small enough and using the smooth coarea formula in the place of the euclidean one, following the proof of *(i)* in Proposition 6.2.1 we can cover the vortex set $\{r_0 \leq \beta_M\}$ with a collection of $n \leq \deg(L)$ geodesic balls $\{B_\sigma(z_k)\}_{k=1}^n$ with small enough radius $\sigma > 0$; namely, $\{r_0 \leq \beta_M\} \subset \bigcup_{k=1}^n B_\sigma(z_k)$. Now since $n \leq \deg(L) \leq \frac{1}{4\pi}\mathrm{vol}(M)$, we can take σ, β_M small enough such that the complement contains a geodesic ball. Precisely, there exists a point x_0 and a radius $\rho > c_M$ such that $r_0 > \beta_M$ on $B_{2\rho}(x_0)$. This proves *(i)*.

Now we prove part *(ii)* in each ball $B_\sigma(z_k)$. Consider the zeros of r_0 inside $B_\sigma(z_k)$ and name them $x_1, \ldots, x_{n_k}$ (without loss of generality). Now consider the function $\tilde{\omega}_k(x) = \Pi_{j=1}^{n_k} e^{-G_{x_j}(x)}$, where $G_p(x)$ is the Green's function for the ball $B_{2\sigma}(z_k)$ centered on p. We see that:

$$-\Delta_g \log(\tilde{\omega}) = -\sum_{j=1}^{n_k} \delta_{x_j} \text{ inside } B_\sigma(z_k)\,.$$

Subtracting the above display from (9.1) and using the maximum principle, we can see that $\|\log(\frac{\tilde{\omega}_k}{r_0})\|_{L^\infty(B_\sigma(z_k))} \leq C_M$. By [56, eq (1.1)] (see (3.5) and the paragraph after) we can see that $\tilde{\omega}_k$ is locally comparable to $\Pi_{j=1}^{n_k} d(x,x_j)$. Then we put together this estimate for all $k = 1, \ldots, n$ and use the global bound $|u_0| \leq 1$ together with compactness of M to obtain *(ii)*.

Item *(iii)* simply follows by compactness and choosing c_M small enough. □

9.2 Stability for regular pairs

The goal now is to prove the stability for regular enough pairs (analogous to Theorem 7.0.1), in the following theorem:

Theorem 9.2.1. *For any compact smooth Riemann surface (M,g) and $\Lambda \geq 1$ there exists $\eta_{\Lambda,M}, C_{\Lambda,M} > 0$ with the following property: Let $L \longrightarrow M$ be a Hermitian line bundle on M with $0 \leq \deg(L) \leq \frac{1}{4\pi}\mathrm{vol}(M)$ and let (u,∇) be a pair such that:*

(i) $\star d((\frac{u}{|u|})^*(d\theta)) = 2\pi \sum_{k=1}^{\deg(L)} \delta_{x_k}$,

(ii) $E(u,\nabla) - 2\pi\deg(L) \leq \eta^2_{\Lambda,M}$,

(iii) $\Lambda^{-1}|u_0| \leq |u| \leq \Lambda|u_0|$,

where (u_0,∇_0) is the solution of the vortex equations (6.4) on $L \longrightarrow M$ with the zero set $x_1,\ldots,x_{\deg(L)} \in M$ (counted with multiplicity). Then:

$$\||u| - |u_0|\|^2_{L^2(M)} + \|F_\nabla - F_{\nabla_0}\|^2_{L^2(M)} \leq C_{\Lambda,M}\left[E(u,\nabla) - 2\pi\deg(L)\right]\,.$$

First we prove the analogous statement to Proposition 7.1.1. We need to subtract a geodesic ball from the manifold, since the estimates of Part II only work on surfaces with boundary. More precisely, there are no non-constant weights that satisfy Definition 4.0.1 on compact surfaces, so the statements of Part II are empty on a compact manifold.

Proposition 9.2.2. *Let (M, g) be a Riemannian surface. Then for any $\Lambda > 1$ and integer $n > 0$ there exists a constant $C_{n,\Lambda,M}$ with the following property. Let ω be a weight as in (3.5) with integer powers. Precisely:*

$$\omega = \Pi_{k=1}^{n} d(x, x_k) \text{ for a collection } \{x_k\}_{k=1}^{n} \subset M \text{ counted with multiplicity}.$$

Moreover let $B_\rho(x_0)$ be a geodesic ball with radius $c_M \leq \rho \leq 2c_M$. Then for any compactly supported function $h \in C_c^\infty(M \backslash B_\rho(x_0))$ and one-form $B \in C_c^\infty(\bigwedge^1 M \backslash B_\rho(x_0))$ the following weighted inequality holds:

$$\int_{M \backslash B_\rho(x_0)} \omega^2 |h|^2 \leq C_{n,\Lambda,M} \int_{M \backslash B_\rho(x_0)} \left[\omega^2 | \star dh + B|^2 + | \star dB + V(x) h|^2 \right] ,$$

provided that $0 \leq V(x) \leq \Lambda \omega(x)^{1+\frac{1}{n}}$.

Proof. Since the estimates in Part II have universal constants and work on arbitrary surfaces with boundary, we can apply the proof of Proposition 7.1.1 almost verbatim. The only difference is that we also need to keep track of a sequence of geodesic balls $B_{\rho_k}(x_{0^k})$ subtracted from the manifold. However since the radius is bounded above and below away from zero ($c_M \leq \rho_k \leq 2c_M$) and the manifold M is compact, we can extract a sub-sequential limit to some $M \backslash B_{\rho_\infty}(x_{0^\infty})$ with $c_M \leq \rho_{0^\infty} \leq 2c_M$. The rest of the proof is unchanged. □

In the rest of this subsection we work to patch the estimates of the subtracted ball to the rest of the manifold and conclude the stability for regular enough pairs.

Proof of Theorem 9.2.1. Up to conjugating u, assume that $\deg(L) \geq 0$. Now observe that, after identifying $\nabla : d - iA$ and $u = re^{i\theta}$ for some $\theta : M \to S^1$ and one-form $A \in \bigwedge^1(M)$ the energy has the following form:

$$E(u, \nabla) - 2\pi \deg(L) = \int_M \left[r^2 | \star d \log(r) + A - d\theta|^2 + | \star dA - \frac{1 - r^2}{2}|^2 \right] .$$

Then by the assumption *(i)* we can gauge fix such that $\frac{u}{|u|} = \frac{u_0}{|u_0|}$, namely u, u_0 have equal phase. Then naming $h = \log(\frac{r}{r_0})$ for $r_0 = |u_0|$ and $B = A - A_0$ for $\nabla_0 : d - iA_0$, we see that:

$$E(u, \nabla) - 2\pi \deg(L) = \int_M \left[r^2 | \star dh + B|^2 + | \star dB + V(x)h|^2 \right] ,$$

where $V(x) = r_0^2 \frac{e^{2h} - 1}{2h}$. By assumption *(iii)* we can see that $\|h\|_{L^\infty(M)} \leq \log(\Lambda)$, hence $C_\Lambda^{-1} r_0^2 \leq V(x) \leq C_\Lambda r_0^2$. Now we proceed similar to the proof of Theorem 7.0.1. In fact, we prove that as a consequence of Proposition 9.2.2 and the Poincaré inequality on small geodesic balls, the discrepancy cannot concentrate on the subtracted ball. In fact we aim to prove that there exists some constant C_M such that:

$$\int_M r^2 |h|^2 \leq C_M \int_M \left[r^2 | \star dh + B|^2 + | \star dB + V(x)h|^2 \right] .$$

Arguing by contradiction, assume there is a sequence $\{r_k, h_k, B_k, V_k\}_{k=1}^\infty$ satisfying the assumptions, such that:

$$\int_M r_k^2 |h_k|^2 = 1 ,$$
$$\eta_k^2 = \int_M \left[r_k^2 | \star dh_k + B_k|^2 + | \star dB_k + V_k(x)h_k|^2 \right] \to 0 .$$

Defining $e_1 = \star dh_k + B_k$ and $e_2 = \star dB_k + V_k(x)h_k$ the above display takes the following form:

$$-\Delta_g h_k + V_k(x)h_k = \star d(\frac{e_1}{r_k}) + e_2 \text{ with } \|e_1\|^2_{L^2(M)} + \|e_2\|^2_{L^2(M)} \leq C\eta_k^2 \,. \tag{9.1}$$

Now take the geodesic ball $B_{\rho_k}(x_k)$ as in *(i)* Proposition 9.1.1 and define the two cut-off functions $0 \leq \psi_k, \phi_k \leq 1$ as follows:

$$\begin{cases} \phi_k = 1 \text{ on } B_{\rho_k/2}(x_k)\,, \\ \phi_k = 0 \text{ on } M\backslash B_{\rho_k}(x_k)\,, \\ |d\phi_k| \leq C_M\,. \end{cases} \quad \begin{cases} \psi_k = 1 \text{ on } M\backslash B_{\rho_k/2}(x_k)\,, \\ \psi_k = 0 \text{ on } B_{\rho_k/3}(x_k)\,, \\ |d\psi_k| \leq C_M\,. \end{cases}$$

First we test (9.1) with $\phi_k h_k$, integrate by parts and use the estimates for r_k on $B_{\rho_k}(x_k)$ with Young's inequality to see that:

$$\int_{B_{\rho_k/2}(x_k)} |h_k|^2 + |dh_k|^2 \leq C_{M,\Lambda}\eta_k^2 + \int_{B_{\rho_k}(x_k)} |h_k|^2\,. \tag{9.2}$$

Now we apply Proposition 9.2.2 for $\psi_k h_k, \psi_k B_k$ to see that (using assumption *(iii)* and *(iii)* in Proposition 9.1.1):

$$\begin{aligned}
&\int_{M\backslash B_{\rho_k/2}(x_k)} r_k^2|h_k|^2 \leq \int_{M\backslash B_{\rho_k/3}(x_k)} r_k^2|\psi_k h_k|^2 \\
&\leq C_{M,\Lambda}\int_{M\backslash B_{\rho_k/3}(x_k)} \left[r_k^2| \star d(\psi_k h_k) + \psi_k B_k|^2 + | \star d(\psi_k B_k) + V_k(x)(\psi_k h_k)|^2\right] \\
&\leq C_{M,\Lambda}\left[\eta_k^2 + \int_{B_{\rho_k/2}(x_k)} |h_k|^2 + |dh_k|^2\right] \leq_{(9.2)} C_{M,\Lambda}\left[\eta_k^2 + \int_{B_{\rho_k}(x_k)} |h_k|^2\right].
\end{aligned}$$

Notice that on $B_{\rho_k}(x_k)$ we have $r_k \geq \beta_M > 0$. Combining the above display with $\|r_k h_k\|_{L^2(M)} = 1$

we see that for large enough k:

$$\int_{B_{\rho_k}(x_k)} |h_k|^2 \geq C_{M,\Lambda} > 0\,.$$

Similarly testing (9.1) with $\chi_k h_k$ such that $\chi_k = 1$ on $B_{\rho_k}(x_k)$ and $\chi_k = 0$ on $M\backslash B_{2\rho_k}(x_k)$, we see that:

$$\int_{B_{\rho_k}(x_k)} |h_k| + |dh_k|^2 \leq C_{M,\Lambda}\,.$$

Note that by *(iv)* in Proposition 9.1.1 and $c_M \leq \rho_k \leq 2c_M$ the geodesic balls $B_{\rho_k}(x_k)$ are uniformly bi-Lipschitz to the unit disk in the euclidean plane. Combining the last two displays, using Banach–Alagolu, Rellich–Kondrachov theorem together we the bounds on the radius, we may extract a convergent sub-sequence in $W^{1,2}$. Moreover the weak limit satisfies (9.1) with $\|e_1\|_{L^2(M)} = \|e_2\|_{L^2(M)} = 0$, testing the equation by h we see that any weak limit of h_k should be 0. However since the L^2 norm of h_k is uniformly bounded below and away from zero, by the strong L^2 convergence, contradiction follows. □

Similar to Corollary 7.1.2 we have that:

Corollary 9.2.3. *As a consequence of the estimates above we also get that there exists a constant $C_{\Lambda,M} > 0$ such that for any $0 < \varepsilon < \frac{1}{N}$:*

$$\int_M \omega^{2+2\varepsilon}|dh|^2 \leq \frac{C_{\Lambda,R,N}}{\varepsilon^2}\int_M \omega^2|\star dh + B|^2 + |\star dB + V(x)h|^2\,.$$

9.3 Selection principle and proof of stability on compact surfaces

Here we find a *regular enough* pair satisfying the assumptions of Theorem 9.2.1 second order close to any nearly minimizing pair $(u, \nabla) \in W^{1,2}(M)$. The proof is adapted with little modification.

Theorem 9.3.1. *Let (M, g) be a compact Riemannian surface. Then there exists constants η_M, C_M, Λ_M with the following property: Let $L \to M$ be a Hermitian line bundle with $0 \leq \deg(L) \leq \frac{1}{4\pi}\mathrm{vol}(M)$. Then for any pair of section and connection $(u, \nabla) \in W^{1,2}(M)$ such that $E(u, \nabla) - 2\pi \deg(L) \leq \eta_M^2$. Then there exists another pair $(\tilde{u}, \tilde{\nabla})$ with the following properties:*

(i) $\|u - \tilde{u}\|^2_{L^2(M)} + \|F_\nabla - F_{\tilde{\nabla}}\|^2_{L^2(M)} \leq C_N \left[E(u, \nabla) - 2\pi \deg(L)\right].$

(ii) $\Lambda_M^{-1}|u_0| \leq |\tilde{u}| \leq \Lambda_M |u_0|$ *for some solution (u_0, ∇_0) with $E(u_0, \nabla_0) = 2\pi \deg(L)$.*

(iii) $2\pi \deg(L) \leq E(\tilde{u}, \tilde{\nabla}) \leq E(u, \nabla)$.

Proof. The proof is essentially the same as Theorem 8.0.1. Similar to Remark 8.1.2 we can assume without loss of generality that $|u| \leq 3$. Then we replace (u, ∇) with the minimizer of the following auxiliary energy:

$$\mathcal{G}_{(u,\nabla)}(u_1, \nabla_1) = E(u_1, \nabla_1) + \|u - u_1\|^2_{L^2(M)} + \|A - A_1\|^2_{L^2(M)}.$$

The existence and regularity follow verbatim as in Lemma 8.1.1 and Lemma 8.2.1 respectively. (In fact the proof of existence is a bit simplified since it is straightforward to prove that the degree passes to the limit) Iterating the process $\deg(L)$ times, we see that we can find a pair $(\tilde{u}, \tilde{\nabla})$ such that:

$$\|\tilde{u} - u\|^2_{L^2(M)} + \|F_{\tilde{\nabla}} - F_\nabla\|^2_{L^2(M)} \leq C_M \left[E(u, \nabla) - 2\pi \deg(L)\right].$$

Moreover, for any simply connected domain $\Omega \Subset M$ there exists a constant $C_{\Omega,M}$ with the following property: After a suitable gauge transformation $(\tilde{u}, \tilde{A}) \longrightarrow (\tilde{u}e^{i\xi}, \tilde{A} + d\xi)$ for $\xi \in W_0^{1,2}(\Omega)$ such that:

$$\Delta_g \xi = d^* \tilde{A} \text{ inside } \Omega \subset M \text{ and } \nabla_\nu \xi = \iota_\nu \tilde{A} \text{ on } \partial\Omega\,,$$

we have the following estimate:

$$\|\tilde{u}e^{i\xi}\|_{C^{\deg(L),\alpha}(\Omega)} \leq C_{M,\Omega}\,.$$

Arguing by contradiction and compactness, using Arzala-Ascoli we can see that if η_M^2 is small enough, then $\tilde{u}e^{i\xi}$ is a C^N perturbation of some solution u_0. Now we take a local chart of $U \subset \Omega$ and map everything onto an euclidean ball; then taking small enough domains and arguing by compactness, we can apply Lemma A.0.1 to conclude as in the proof of Theorem 8.0.1. □

Proof of Theorem 5.2.1. Using Theorem 9.3.1 and Theorem 9.2.1, the proof of Theorem 5.1.1 applies verbatim. □

Proof of Theorem 5.2.2. The proof follows from Corollary 9.2.3, similar to the proof of Theorem 5.1.2 and Theorem 5.1.3. □

Part IV

Decay of Excess for the abelian-Higgs model

Chapter 10

Introduction

10.1 Background on the Allen–Cahn and abelian Higgs models

Area of geometric shapes is one of the oldest geometric functional considered in mathematics. Given an ambient Riemannian manifold (M^n, g) (possibly the flat Euclidean space $\mathbb{R}^n$) and given an integer $1 \leqslant k \leqslant n-1$, one looks for k-dimensional objects, such as k-dimensional submanifolds or singular versions of them, which are *critical points* for the k-area $\mathcal{H}^k$. These are called *minimal submanifolds* (provided they are regular enough, depending on the context).

Besides its intrinsic interest, the study of minimal submanifolds in a given ambient often reveals global topological structure, especially when coupled with curvature information.

These applications motivate a systematic existence and regularity theory of such critical points. In spite of its apparent simplicity, it is notoriously difficult to use the area functional directly in the context of the *calculus of variations*, especially when $k \geqslant 2$. Leaving out a number of very important ways to deal with this problem, such as the approach via parametrizations when $k = 2$, see, e.g., [30, 65–67] among others, area-minimizing *currents* and *sets of finite perimeter* in the context of minimization, [23, 32] and the monographs [33, 72], and the Almgren–Pitts theory involving *varifolds*, [2, 64]. In this paper we focus on the approximation of minimal surfaces as limit of *diffuse* physical energies.

Starting from the pioneering ideas of De Giorgi, Modica [59], Ilmanen [49], and Hutchinson–Tonegawa [48], it was understood that smooth critical points $u : M \to \mathbb{R}$ for the *Allen–Cahn energy*

$$E_\varepsilon(u) := \int_M \left[\varepsilon|du|^2 + \frac{(1-u^2)^2}{4\varepsilon}\right]$$

are effective diffuse approximations of minimal hypersurfaces. The Allen–Cahn functional is a well studied model for phase transitions; a typical critical point u takes values in $[-1, 1]$, with $u \approx \pm 1$ (the pure phases) except in a transition region of thickness $\approx \varepsilon$, where most of the energy concentrates. Roughly speaking, this region is an ε-neighborhood of a minimal hypersurface, which acts as an interface between the two phases, and the energy density decays exponentially fast away from this interface.

This understanding brought a novel, PDE-based way to attack variational problems for the co-dimension-one area [43], which often allows to obtain more refined results compared to other methods [17].

In co-dimension two, similar attempts have been made by looking at the same energy for maps $u : M \to \mathbb{C}$, replacing u with $|u|$ in the second term. This corresponds to a simplified version of the Ginzburg–Landau model of superconductivity, popularized by Bethuel–Brezis–Hélein [9], where one neglects the magnetic field. The asymptotic analysis of this energy is substantially more involved, due to the lack of the aforementioned exponential decay, and brought mixed results: see, for instance, [8, 57] in the positive direction and [63] in the negative one.

On the other hand, including the magnetic field and looking at the so-called *self-dual regime* (also called *critical coupling*), we can consider the alternative energy

$$E_\varepsilon(u, \alpha) := \int_M \left[|du - i\alpha u|^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2} + \varepsilon^2|d\alpha|^2\right].$$

Apart from the different normalization, it differs from the previous energies by an additional variable, the one-form $\alpha \in \Omega^1(M;\mathbb{R})$, which twists the Dirichlet term and appears in the Yang–Mills term $|d\alpha|^2$ (indeed, the latter equals $|F_\nabla|^2$, where F_∇ is the curvature of the connection $\nabla := d - i\alpha$ on the trivial complex line bundle $\mathbb{C} \times M$).

This energy, in this specific self-dual regime (i.e., the choice of constants in front of each term), is well known in gauge theory, where it is often called *$U(1)$-Yang–Mills–Higgs*, or simply *abelian Higgs model*. It received a thorough treatment in dimension 2, with a complete classification of critical planar pairs (u, ∇) of finite energy by Taubes [74, 75]. See also [47] for the case of Riemann surface and [12] for Kähler manifolds. Recently, in [62], Stern and the Pigati developed the asymptotic analysis in arbitrary Riemannian manifolds, obtaining the precise co-dimension-two analogue of the result by Hutchinson–Tonegawa: see Theorem 13.1.1 below. Related facts, including Γ-convergence and the gradient flow convergence to mean curvature flow, have also been verified, by Parise, Stern, and Pigati [60, 61].

Based on some new functional inequalities [44] (in Part II), we obtain a quantitative refinement of the work of Taubes (Part III), who showed (among other facts) that critical pairs on the plane minimize the energy among pairs with the same degree at infinity: namely, in [45] a quantitative *stability* is proved; the precise statement is recalled in Theorem 13.4.1 (for convenience). Together with the main result from [62], this result will be instrumental for the analysis in the present paper.

10.2 Savin's theorem

Since the work of De Giorgi [24] and Allard [2], it is known that almost-flat minimal submanifolds enjoy an *improvement of flatness*, i.e., they become even closer to a plane at smaller scales, in a quantitative way. Iteration of this improvement of flatness is the key mechanism in proving (quantitative) regularity of minimal surfaces. The key analytical fact behind this decay property is the observation that the linearization of the minimal graph equation is the Laplace equation, whose solutions enjoy similar decay properties.

A related question, in the spirit of the classical Liouville theorem, is whether globally defined objects should be planar. The famous *Bernstein's conjecture* predicts that this is always true for minimal graphs $\mathbb{R}^{n-1} \to \mathbb{R}$, which are automatically (locally) area-minimizing hypersurfaces. In view of the improvement of flatness, this question quickly reduces to understanding whether any blow-down is necessarily a hyperplane. Bernstein's question was answered affirmatively by the works of Fleming, De Giorgi, Almgren, and Simons for $n \leqslant 8$, while Bombieri–De Giorgi–Giusti produced a counterexample for $n = 9$, whose blow-down corresponds to the Simons cone, in [11].

By analogy, De Giorgi conjectured that critical points $u : \mathbb{R}^n \to \mathbb{R}$ of the Allen–Cahn energy with $\frac{\partial u}{\partial x_n} > 0$ (so that level sets are graphs) are just rotations of a one-dimensional solution $u = u(x_n)$, at least when $n \leqslant 8$. The question has been solved by Ghoussoub–Gui for $n = 2$, in [42], by Ambrosio–Cabré for $n = 3$, in [4], and by Barlow–Bass–Gui under additional regularity for the level sets, in [6]. Finally, in [70] Savin settled the conjecture for all $n \leqslant 8$ under the assumption that $u(x', x_n) \to \pm 1$ as $x_n \to \pm\infty$, for any fixed $x' \in \mathbb{R}^{n-1}$. In fact, his main contribution could be phrased as follows.

Theorem 10.2.1 (Savin's theorem)**.** *A local minimizer u for Allen–Cahn enjoys improvement of flatness. In particular, if any blow-down is a hyperplane, then the blow-down is unique.*

Here the blow-downs can be understood in terms of energy concentration, or by looking at the blow-downs of the zero set $\{u = 0\}$ with respect to the (local) Hausdorff convergence of sets.

The previous statement implies the resolution of De Giorgi's conjecture for $n \leqslant 8$, with the extra assumption mentioned above. Indeed, it is known that this condition, together with $\frac{\partial u}{\partial x_n} > 0$, implies that u is a local minimizer; moreover, any blow-down gives a vertical area-minimizing cone in $\mathbb{R}^n$, hence an area-minimizing cone in $\mathbb{R}^{n-1}$, which is known to be necessarily a hyperplane for $n \leqslant 8$.

In other words, uniqueness of the blow-down relies on two ingredients: improvement of flatness and a classification of blow-downs. While the second one can be directly exported from the setting of minimal hypersurfaces, the first one needs to be proved *before* passing to the limit $\varepsilon \to 0$, and this is the difficult part settled by Savin.

Finally, using the maximum principle (see, e.g., [7, 31]), one can deduce the following.

Corollary 10.2.2. *Under the previous assumptions, u is one-dimensional.*

As for minimal graphs, De Giorgi's conjecture (even with the extra assumption used by Savin) is false for $n \geqslant 9$: a counterexample has been constructed by Del Pino–Kowalczyk–Wei, in [29].

Savin's approach uses viscosity techniques, resembling the Krylov–Safanov theory in spirit. In particular, while his groundbreaking methods have a wide range of applicability, even beyond variational equations, it is not always clear how one can extend these techniques to the vectorial setting, where the maximum principle does not apply; see however [27, 69].

Recently, Wang [80] obtained a variational proof of Savin's theorem, following the strategy of Allard's proof of excess decay for stationary varifolds. Wang's paper has been the starting point for our investigation of the regularity properties of the zero set of solutions of the Yang–Mills–Higgs equations.

10.3 Main results

We consider the energy

$$E_\varepsilon := \int e_\varepsilon(u, \nabla), \quad e_\varepsilon(u, \nabla) := |\nabla u|^2 + \frac{(1 - |u|^2)^2}{4\varepsilon^2} + \varepsilon^2 |F_\nabla|^2.$$

Note that E_ε is just a rescaling of E_1, for $\varepsilon > 0$. The main result of the paper could be summarized as follows.

Theorem 10.3.1. *Savin's result, as stated in Theorem 10.2.1, holds for critical pairs (u, ∇) for E_1, in any dimension $n \geqslant 2$.*

The following is the precise statement of the excess decay for critical points.

Theorem 10.3.2 (Tilt-excess decay). *For any $n \geq 3$ and small enough $0 < \rho \leq \rho_0(n)$, there exist constants $\varepsilon_0(n, \rho), \tau_0(n, \rho)$ such that the following holds. Let (u, ∇) be a critical point for E_ε on the unit ball $B_1^n \subset \mathbb{R}^n$, with $\varepsilon \leq \varepsilon_0$, $u(0) = 0$, and the energy bound*

$$\frac{1}{|B_1^{n-2}|} \int_{B_1^n} e_\varepsilon(u, \nabla) \leq 2\pi + \tau_0. \tag{10.1}$$

Then at least one of the following statements is true: either

$$\boldsymbol{E}_1(u, \nabla, B_\rho^n, \overline{S}) \leq C\rho^2 \boldsymbol{E}_1(u, \nabla, B_1^n, S),$$

for some $(n-2)$-plane $\overline{S}$ with $\|P_{\overline{S}} - P_S\| \leq C\sqrt{\boldsymbol{E}_1(u, \nabla, B_1^n, S)}$, where P_S is the orthogonal projection onto S, the plane minimizing $\boldsymbol{E}(u, \nabla, B_\rho^n, \cdot)$, or

$$\boldsymbol{E}_1(u, \nabla, B_1^n, S) \leq \max\{C\varepsilon^2 |\log \boldsymbol{E}|^2 \sqrt{\boldsymbol{E}}, e^{-K/\varepsilon}\},$$

where $\boldsymbol{E} = \boldsymbol{E}(u, \nabla, B_1^n, S)$ and $C = C(n)$, $K = K(n)$ are independent of ρ.

Remark 10.3.3. To be precise, we assume also the pointwise bounds (13.1)–(13.2), which are automatically true if (u, ∇) is a critical pair on $\mathbb{R}^n$ with energy growth $O(R^{n-2})$ on B_R^n.

Here $\mathbf{E}$ is the *excess*, defined in (14.1) below, which naturally splits into two parts, $\mathbf{E}_1$ and $\mathbf{E}_2$, measuring how far a solution is from being two-dimensional and from solving the first order *vortex equations*, respectively. We also note that $\mathbf{E}_1$ parallels the notion of excess in the theory of varifolds and does not depend on the orientation, while $\mathbf{E}$ sees the orientation and should be thought of as the stronger notion of excess in the setting of currents. While in principle the previous result establishes a quantitative decay only for $\mathbf{E}_1$, it is enough to obtain the following.

Corollary 10.3.4. *If (u, ∇) is an entire critical point on $\mathbb{R}^n$, with*

$$0 < \lim_{R\to\infty} \frac{1}{|B_R^{n-2}|} \int_{B_R^n} e_\varepsilon(u,\nabla) \leq 2\pi + \tau_0(n),$$

then this limit is 2π and the blow-down is a unique plane.

The previous limit always exists by the monotonicity formula for E_ε, see [62]. By a simple compactness argument and Allard's theorem, it is easy to see that the assumption guarantees that any blow-down is an $(n-2)$-dimensional plane. The key assertion is that, in view of improvement of flatness, the blow-down is *unique*.

Another simple consequence of the techniques is the following fact, a diffuse version of the $C^{1,\alpha}$ regularity of minimal graphs.

Theorem 10.3.5. *Let (u, ∇) be a critical point for E_ε as above. Given $\alpha \in [0, 1)$ and $\gamma > 0$, if $\varepsilon \leqslant \varepsilon_0(n, \alpha, \gamma)$ and $\tau_0 \leqslant \tau_0(n, \alpha, \gamma)$ then the vorticity set $\{|u| \leqslant \frac{3}{4}\} \cap B^n_{1/2}$ is contained in a $C(n, \alpha, \gamma)\varepsilon^{1/(1+\alpha)}$-neighborhood of the graph of a function*

$$f : B_1^{n-2} \to \mathbb{R}^2$$

with $\|f\|_{C^{1,\alpha}} \leqslant \gamma$.

Differently from the co-dimension one setting, where uniqueness of the blow-down (with multiplicity one) implies via the maximum principle that u is one-dimensional, at the present time we are not able to conclude that, in the setting of Corollary 10.3.4, the solution u is two-dimensional. Here we formulate the following variant of the *Gibbons conjecture.*

Conjecture 10.3.6. *An entire critical point (u, ∇) on $\mathbb{R}^n$ satisfying*

$$\lim_{R\to\infty} \frac{1}{|B_R^{n-2}|} \int_{B_R^n} e_\varepsilon(u, \nabla) = 2\pi$$

and, writing any $x \in \mathbb{R}^n$ as $x = (y, z) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$, also

$$\lim_{|y|\to\infty} |u(y, z)| = 1, \quad \text{uniformly in } z,$$

is necessarily two-dimensional, i.e., it is the pullback through the projection $\mathbb{R}^n \to \mathbb{R}^2$ of the standard solution in $\mathbb{R}^2$ with degree ± 1, up to translation and change of gauge.

It is interesting to note that, if we allow for multiplicity higher than one in the blow-down, this conjecture (with the appropriate energy assumption) does *not* hold for the non-magnetic Ginzburg–Landau energy mentioned before, see [22]. It is not clear if such rigidity with higher multiplicity should be expected for the energy considered in the present work. On the other hand, our excess decay is strong enough to give an affirmative answer up to dimension 4. With a more involved argument, we are able to settle it also for local minimizers in all dimensions $n \geq 2$, thus obtaining a full analogue of Savin's theorem.

Theorem 10.3.7. *The previous conjecture holds for critical points in dimension $2 \leqslant n \leqslant 4$, as well as for local minimizers in all dimensions $n \geqslant 2$, even without the second assumption that $\lim_{|y|\to\infty} |u(y, z)| = 1$ uniformly in z: the pair (u, ∇) is two-dimensional, up to rotation and change of gauge.*

The techniques used in this paper resemble those of [80] at several places. However, there are several key differences which require substantially new ideas. For instance, in order to construct the Lipschitz approximation, Wang uses a generic level set of u. The fact that typical level sets share effectively properties of minimal hypersurfaces is often used in [80], as well as in [17, 49, 77] and many other works in the Allen–Cahn setting. For the abelian Higgs model, level sets of u

can be arbitrarily irregular, due to gauge invariance; while we can always pass to a local Coulomb gauge, we do not expect such effective properties of typical preimages of u.

Rather, in the present setting, we rely on the results from [45] in order to control in a fine way the behavior of u on many (but not all) two-dimensional slices perpendicular to the reference plane. For instance, we are able to bound the distance of the actual zero set from a certain function giving the "center of mass" of each slice, which is used as a Lipschitz approximation and allows to derive a Caccioppoli-type inequality.

In the case of minimizers, this refined control also allows us to deform a nearly flat minimizing pair (u, ∇) in the interior to gain a *stronger* decay of the excess. This deformation process also requires a very involved gauge fixing argument, since generically (u, ∇) could be very irregular in an arbitrary gauge. The following theorem is the precise statement of the improved tilt-excess decay that we obtain for minimizers. Note that in the statement below, β can indeed be any power. This is fundamental for proving Theorem 10.3.7 for minimizers, where we need to take $\beta = n - 2$.

Theorem 10.3.8. *For any $\beta > 0$ and small enough $0 < \rho \leqslant \rho_0(n, \beta)$ there exist $\tau_0(n, \beta, \rho) > 0$, $\varepsilon_0(n, \beta, \rho) > 0$ with the following property. Let (u, ∇) be a local minimizer of E_ε in B_1^n with $\varepsilon \leq \varepsilon_0$ and $u(0) = 0$ such that*

$$\frac{1}{|B_1^{n-2}|} \int_{B_1^n} e_\varepsilon(u, \nabla) \leq 2\pi + \tau_0,$$

and let S minimize $\boldsymbol{E}(u, \nabla, B_1^n, S)$. Then, after a suitable rotation, at least one of the following statements is true: either

$$\boldsymbol{E}(u, \nabla, B_\rho^n, \overline{S}) \leq C\rho^2 \boldsymbol{E}(u, \nabla, B_1^n, S),$$

for some new oriented $(n-2)$-plane $\overline{S}$ with $\|P_{\overline{S}} - P_S\| \leq C\sqrt{\boldsymbol{E}}$, or

$$\boldsymbol{E}(u, \nabla, B_1^n, \mathbb{R}^{n-2}) \leq \varepsilon^{\beta},$$

where $C = C(n, \beta)$ is independent of ρ.

Then, by taking $\beta \geqslant n-2$, we obtain a direct proof of Theorem 10.3.7 in the case of minimizers.

Chapter 11

Basic definitions

While we work on the trivial Hermitian line bundle over the Euclidean space $\mathbb{R}^n$, it is worth to recall the definition of Hermitian line bundle over a general manifold.

Definition 11.0.1. A *Hermitian line bundle* over a smooth manifold M is a complex line bundle $L \to M$ (i.e., a complex vector bundle with typical fiber $\mathbb{C}$) equipped with a *Hermitian metric*, whose real part will be denoted by $\langle \cdot, \cdot \rangle$; thus, for any two smooth sections $s, t \in \Gamma(L)$, the function $p \mapsto \langle s(p), t(p) \rangle$ is smooth and real-valued, and satisfies $\langle is(p), it(p) \rangle = \langle s(p), t(p) \rangle = \langle t(p), s(p) \rangle$.

Definition 11.0.2. A *metric connection* is a map ∇ which assigns to each vector field $\xi \in \Gamma(TM)$ an endomorphism $\nabla_\xi : \Gamma(L) \to \Gamma(L)$ with the following properties:

(i) $\nabla_{\xi+\eta} s = \nabla_\xi s + \nabla_\eta s$;

(ii) $\nabla_{\phi\xi} s = \phi \nabla_\xi s$;

(iii) $\nabla_\xi(\phi s) = (\xi\phi)s + \phi\nabla_\xi s$;

(iv) $\xi(\langle s, t \rangle) = \langle \nabla_\xi s, t \rangle + \langle s, \nabla_\xi t \rangle$,

for any sections $s, t \in \Gamma(L)$, vector fields $\xi, \eta \in \Gamma(TM)$, and function $\phi \in C^\infty(M)$.

On the trivial bundle $L = \mathbb{C} \times M$, we can always write a metric connection ∇ as

$$\nabla = d - i\alpha,$$

for a real-valued one-form, meaning that $\nabla_\xi s = ds(\xi) - i\alpha(\xi)s$.

In general, for two vector fields ξ and η, typically ∇_ξ and ∇_η do not commute, meaning that the connection has nontrivial *curvature*. Formally, the curvature F_∇ is given by

$$F_\nabla(\xi, \eta)(s) = [\nabla_\xi, \nabla_\eta]s - \nabla_{[\xi,\eta]}s. \tag{11.1}$$

A simple computation shows that F_∇ is a two-form with values in the Lie algebra of $U(1)$, i.e., in imaginary numbers; we will sometimes use the real-valued two-form ω given by

$$F_\nabla(\xi, \eta)(s) =: -i\omega(\xi, \eta)s. \tag{11.2}$$

On the trivial bundle, if $\nabla = d - i\alpha$ then we simply have

$$\omega = d\alpha.$$

We will use the inner product on two-forms induced by the following quadratic form:

$$|\omega|^2 = \sum_{1 \le j < k \le n} |\omega(e_j, e_k)|^2,$$

where $\{e_k\}_{k=1}^n$ is a local orthonormal frame for TM.

Chapter 12

The $U(1)$-Yang–Mills–Higgs equations

For a section $u \in \Gamma(L)$ and a (metric) connection ∇ on a Hermitian line bundle $L \to M$ over a smooth Riemannian manifold (M, g), given a parameter $\varepsilon > 0$, we define the $U(1)$-Yang–Mills–Higgs energy as

$$E_\varepsilon(u, \nabla) := \int_M \left[|\nabla u|^2 + \varepsilon^2 |F_\nabla|^2 + \frac{(1 - |u|^2)^2}{4\varepsilon^2} \right], \tag{12.1}$$

where F_∇ is the curvature of ∇ and $|F_\nabla|$ is defined to be $|\omega|$ (with ω as in (11.2)). Equivalently, on the trivial bundle, for any section u (viewed as a function $M \to \mathbb{C}$) and connection $\nabla = d - i\alpha$ we have

$$E_\varepsilon(u, \nabla = d - i\alpha) = \int_M \left[|du - iu\alpha|^2 + \varepsilon^2 |d\alpha|^2 + \frac{(1 - |u|^2)^2}{4\varepsilon^2} \right].$$

A smooth pair (u, ∇) gives a critical point for the Yang–Mills–Higgs energy if and only if it satisfies the system of partial differential equations:

$$\nabla^*\nabla u = \frac{1}{2\varepsilon^2}(1-|u|^2)u, \tag{12.2}$$

$$\varepsilon^2 d^*\omega = \langle \nabla u, iu\rangle, \tag{12.3}$$

where ∇^* is the adjoint of ∇, while d^* is the adjoint of $d : \Omega^1(M) \to \Omega^2(M)$, given by

$$(d^*\omega)(e_k) = -\sum_{j=1}^{n}(\nabla_{e_j}\omega)(e_j, e_k)$$

for some (and hence any) orthonormal frame $\{e_j\}$.

We now recall some Bochner-type identities from [62, Sections 2–3]. Since ω is a closed two-form, after taking the exterior derivative in (12.3) we get

$$\varepsilon^2\Delta_H\omega + |u|^2\omega = \psi(u), \tag{12.4}$$

where $\Delta_H = dd^* + d^*d$ is the Hodge Laplacian and

$$\psi(u)(e_j, e_k) := 2\langle i\nabla_{e_j}u, \nabla_{e_k}u\rangle. \tag{12.5}$$

One easily sees that the modulus $|u|^2$ satisfies the equation

$$\Delta\frac{1}{2}|u|^2 = |\nabla u|^2 - \frac{|u|^2}{2\varepsilon^2}(1-|u|^2). \tag{12.6}$$

We also recall the following Bochner identity for $|\nabla u|^2$:

$$\Delta\frac{1}{2}|\nabla u|^2 = |\nabla^2 u|^2 + \frac{1}{2\varepsilon^2}(3|u|^2-1)|\nabla u|^2 - 2\langle\omega, \psi(u)\rangle + \mathcal{R}_1(\nabla u, \nabla u), \tag{12.7}$$

where $\mathcal{R}_1 = \mathrm{Ric}(e_j, e_k)\langle\nabla_{e_j}u, \nabla_{e_k}u\rangle$ and $\nabla^2_{e_j,e_k}u = \nabla_{e_j}(\nabla_{e_k}u)$.

Next, we define the gauge-invariant Jacobian, which plays an important role in the Γ-convergence theory [60], similar to the classical Jacobian in the Γ-convergence for the Ginzburg–Landau energy with no magnetic field, see [1, 8, 53]. It is the two-form given by

$$J(u,\nabla) := \psi(u) + (1-|u|^2)\omega. \tag{12.8}$$

We have the trivial pointwise bound

$$|J(u,\nabla)| \leq e_\varepsilon(u,\nabla), \tag{12.9}$$

where $e_\varepsilon(u,\nabla)$ is the integrand in (12.1).

We define Γ_ε to be the dual current to the Jacobian, formally identified by the duality formula

$$\langle \Gamma_\varepsilon, \xi \rangle = \frac{1}{2\pi}\int_M J(u,\nabla) \wedge \xi, \tag{12.10}$$

for any $(n-2)$-form $\xi \in \Omega^{n-2}(M)$. Note that by (12.4) the Jacobian can be written as

$$J(u,\nabla) = \omega + \varepsilon^2 \Delta_H \omega.$$

In particular, this shows that the gauge-invariant Jacobian is a closed two-form. This, in duality, implies that $\partial\Gamma_\varepsilon = 0$, i.e., Γ_ε is an $(n-2)$-dimensional cycle.

Chapter 13

Preliminary estimates

13.1 The energy concentration set

It was proved by the Pigati and Stern in [62] that, for a sequence $(u_\varepsilon, \nabla_\varepsilon)$ with $\varepsilon \to 0$, one can extract a subsequence such that the energy density converges to (the weight of) a stationary integer-rectifiable $(n-2)$-varifold. We restate the main result of [62] in the following theorem, see [62, eq. (6.35)] for the conclusion on the Jacobian.

Theorem 13.1.1 (The varifold limit)**.** *Let $L \to M$ be a Hermitian line bundle over a closed, oriented Riemannian manifold (M^n, g) of dimension $n \geq 2$ and let $(u_\varepsilon, \nabla_\varepsilon)$ be a family of critical points of E_ε, satisfying the uniform energy bound*

$$E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \leq \Lambda < \infty.$$

Then, as $\varepsilon \to 0$, the energy measures

$$\mu_\varepsilon = \frac{1}{2\pi} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \operatorname{vol}_g,$$

converge subsequentially, in duality with $C^0(M)$, to a measure μ which is the weight of a stationary *integral* $(n-2)$*-varifold* V*. Also, for all* $0 \le \delta < 1$*,*

$$\operatorname{spt}(V) = \lim_{\varepsilon \to 0} \{|u_\varepsilon| \le \delta\},$$

in the Hausdorff topology. The $(n-2)$*-currents dual to the curvature forms* $\frac{1}{2\pi}\omega_\varepsilon$ *and Jacobians* $\frac{1}{2\pi}J(u_\varepsilon, \nabla_\varepsilon)$ *converge subsequentially to the same limit, an integral cycle* Γ *with* $|\Gamma| \le \mu$*.*

Remark 13.1.2. The previous result admits a local version, proved in the same way (assuming the bounds (13.1) and (13.2) below, which in the closed case follow from the maximum principle): assume that we have an increasing sequence of open sets $U_\varepsilon \subseteq \mathbb{R}^n$ and a sequence of smooth pairs $(u_\varepsilon, \nabla_\varepsilon)$, each defined on the trivial bundle $\mathbb{C} \times U_\varepsilon$ and critical for E_ε; if we have

$$\limsup_{\varepsilon \to 0} \int_K e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) < \infty$$

for any compact subset $K \subset U := \bigcup_\varepsilon U_\varepsilon$, as well as (13.1)–(13.2), then there exist a limiting varifold V and a limiting cycle Γ satisfying the same conclusions as above (up to a subsequence).

We will use the above theorem (in its local version) in several soft arguments by compactness and contradiction; in particular, we will use it to obtain information for any blow-down limit of an entire solution.

13.2 Modica-type bounds and exponential decay

Actually, [62] contains some additional information which will be used frequently in the paper, including a Modica-type bound which was first proved in dimension two in [51, Theorem III.8.1]. We record the following propositions in the non-compact case of $M = \mathbb{R}^n$, with the trivial bundle $L = \mathbb{C} \times \mathbb{R}^n$.

Proposition 13.2.1. *A critical point (u, ∇) for E_ε, on the trivial bundle on $\mathbb{R}^n$, satisfies*

$$|u| \leqslant 1 \tag{13.1}$$

everywhere.

Proof. The bound $|u| \leqslant 1 + C(n)\varepsilon^2$ on the unit ball $B_1(0)$ can be shown as in [61, Proposition A.2], and the claim follows by scaling. □

Proposition 13.2.2 (Modica-type bounds)**.** *Assuming also that the energy on a ball B_R is $O(R^{n-2})$ for R large enough, we have the pointwise bounds*

$$\varepsilon |F_\nabla| \leq \frac{1 - |u|^2}{2\varepsilon}, \quad |\nabla u| \leq \frac{1 - |u|^2}{\varepsilon}. \tag{13.2}$$

Proof. The proof is essentially the same as in [62]; however, in the Euclidean space, the Modica-type bound has no error terms. First, define ξ_ε to be the *discrepancy*:

$$\xi := \varepsilon |F_\nabla| - \frac{1 - |u|^2}{2\varepsilon}. \tag{13.3}$$

Arguing as in [62, Section 3], we see that

$$\Delta \xi \geq \frac{|u|^2}{\varepsilon^2} \xi. \tag{13.4}$$

For the positive part ξ^+, this immediately implies that

$$\Delta \xi^+ \geqslant 0$$

in the distributional sense, i.e., ξ is subharmonic. Under the energy growth assumption, we have

$$\int_{B_R(0)} |\xi| = O(R^{n-1}),$$

which gives $\xi^+ \equiv 0$, as claimed.

For the second bound, proceeding as in [62, eqs. (5.5)–(5.6)], we check that

$$w := |\nabla u| - \frac{1-|u|^2}{\varepsilon}$$

satisfies

$$\Delta w \geq \frac{|u|^2}{\varepsilon^2} w + \frac{1}{\varepsilon}\left(w + \frac{1-|u|^2}{\varepsilon}\right)\left(2w + \frac{1-|u|^2}{2\varepsilon}\right).$$

Again, this implies that w^+ is subharmonic, and hence $w^+ \equiv 0$. $\square$

We also record the following exponential decay of energy, which plays a key role in the paper.

Proposition 13.2.3 (Exponential decay away from the vorticity set)**.** *There exist constants $K(n) > 0$ and $C(n) > 0$ such that, defining $Z := \{|u| \leq \frac{3}{4}\}$ and $r(p) := \mathrm{dist}(p, Z)$, we have*

$$e_\varepsilon(u, \nabla) \leq C \frac{e^{-Kr(p)/\varepsilon}}{\varepsilon^2}. \tag{13.5}$$

Proof. As in [62, Corollary 5.2], we compute that on $\mathbb{R}^n \setminus Z$ we have

$$\Delta \frac{1-|u|^2}{2} \geqslant \frac{1-|u|^2}{4\varepsilon^2}.$$

Exponential decay now follows as in [62, Proposition 5.3], using also the previous Modica-type bounds. $\square$

13.3 Inner variations and monotonicity

In this section we recall the inner variation formulas for critical points. With respect to any orthonormal basis $\{e_k\}_{k=1}^n$ for TM, we define the $(0,2)$-tensors $\nabla u^*\nabla u$ and $\omega^*\omega$ by

$$(\nabla u^*\nabla u)(e_j,e_k) := \langle \nabla_{e_j}u, \nabla_{e_k}u\rangle, \tag{13.1}$$

$$\omega^*\omega(e_i,e_j) := \sum_{k=1}^{n} \omega(e_i,e_k)\omega(e_j,e_k). \tag{13.2}$$

We define the *stress-energy tensor* to be

$$T_\varepsilon(u,\nabla) := e_\varepsilon(u,\nabla) - 2\nabla u^*\nabla u - 2\varepsilon^2\omega^*\omega. \tag{13.3}$$

Then, for any pair (u,∇) satisfying (12.2)–(12.3), the inner variation formula then reads

$$\operatorname{div}(T_\varepsilon(u,\nabla)) = 0, \tag{13.4}$$

meaning that, for any compactly supported vector field X,

$$\int_M \langle T_\varepsilon(u,\nabla), DX\rangle = 0. \tag{13.5}$$

A core tool in the proof of Theorem 13.1.1 is the *monotonicity formula* from [62, Theorem 4.3], which is cleaner in the case of the trivial line bundle $L = \mathbb{C}\times\mathbb{R}^n$ over the flat Euclidean space $M = \mathbb{R}^n$. We state this version of the theorem for convenience and give a short proof.

Proposition 13.3.1 (Monotonicity formula)**.** *Let (u,∇) be a critical point for E_ε on the trivial line* *bundle $L = \mathbb{C} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Then the normalized energy*

$$\tilde{E}_\varepsilon(p,r) := r^{2-n} \int_{B_r(p)} e_\varepsilon(u,\nabla)$$

satisfies

$$\begin{aligned}\frac{d}{dr}\tilde{E}_\varepsilon(p,r) = 2r^{1-n} &\int_{B_r(p)} \left(\frac{(1-|u|^2)^2}{4\varepsilon^2} - \varepsilon^2|\omega|^2\right) \\ &+ 2r^{2-n} \int_{\partial B_r(p)} (|\nabla_\nu u|^2 + \varepsilon^2 |\iota_\nu \omega|^2).\end{aligned} \tag{13.6}$$

Proof. Without loss of generality, assume that $p = 0$. By approximation we can take $X(x) = \mathbf{1}_{B_r(0)} \sum_{k=1}^n x_k e_k$ in (13.5), obtaining

$$\begin{aligned}r\int_{\partial B_r} e_\varepsilon(u,\nabla) = \int_{B_r} (n-2)e_\varepsilon(u,\nabla) &+ 2\int_{B_r}\left(\frac{(1-|u|^2)^2}{4\varepsilon^2} - \varepsilon^2|\omega|^2\right) \\ &+ 2r\int_{\partial B_r}\left(|\nabla_\nu u|^2 + \varepsilon^2|\iota_\nu\omega|^2\right).\end{aligned}$$

Since

$$\begin{aligned}\frac{d}{dr}\tilde{E}_\varepsilon(x,r) &= (2-n)r^{1-n}\int_{B_r} e_\varepsilon(u,\nabla) + r^{2-n}\int_{\partial B_r} e_\varepsilon(u,\nabla) \\ &= 2r^{1-n}\int_{B_r}\left(\frac{(1-|u|^2)^2}{4\varepsilon^2} - \varepsilon^2|\omega|^2\right) \\ &\quad + 2r^{2-n}\int_{\partial B_r}(|\nabla_\nu u|^2 + \varepsilon^2|\iota_\nu\omega|^2),\end{aligned}$$

we obtain the desired conclusion. $\square$

13.4 Quantitative stability in two dimensions and the vortex equations

In this section we record some results regarding the existence, uniqueness, and quantitative stability of critical points for (12.1) in $\mathbb{R}^2$. First of all note that for $\varepsilon = 1$ the energy E_1 of *any* pair (u, ∇) can be written as follows:

$$E(u,\nabla) = \int_{\mathbb{R}^2} \left[|\nabla u|^2 + |F_\nabla|^2 + \frac{(1-|u|^2)^2}{4} \right]$$
$$= 2\pi|N| + \int_{\mathbb{R}^2} |\nabla_1 u \pm i\nabla_2 u|^2 + \left| \star\omega \mp \frac{1-|u|^2}{2} \right|^2, \tag{13.1}$$

where N is the vortex number of (u, ∇), given by

$$N := \frac{1}{2\pi} \int_{\mathbb{R}^2} \star\omega.$$

Thus (u, ∇) is a minimizer of the total energy among pairs with the same vortex number if and only if it satisfies the first-order system of *vortex equations*:

$$\nabla_1 u \pm i\nabla_2 u = 0 \text{ and } \star\,\omega = \pm\frac{1-|u|^2}{2\varepsilon}. \tag{13.2}$$

These are also called *Bogomol'nyi equations* (after [10]) or *self-dual equations*, and arise in many self-dual gauge theories. Taubes, in [74], proved that we can prescribe the zero set $u^{-1}(0) = \{a_1, \dots, a_k\}$: given any finite collection of $k \geqslant 0$ points, counted with multiplicity, there exists a solution (u, ∇) to the vortex equations (with either choice of signs, corresponding to vortex number $N = k$ and $N = -k$, respectively) with this prescribed zero set; moreover, the solution is unique up to change of gauge.

In [45] the previous results we improved to a (sharp) quantitative stability for critical points of E_1. We record these results and this improvement in the following theorem.

Theorem 13.4.1 (Uniqueness and stability in two dimensions Theorem 5.1.1 & Theorem 7.0.1 & Theorem 5.1.3)**.** *On the trivial line bundle over $\mathbb{R}^2$, any critical point (u, ∇) of finite energy for E_1 is actually a minimizer with $E_1(u, \nabla) = 2\pi k \in 2\pi\mathbb{N}$. Moreover, up to change of gauge, any minimizer is uniquely characterized by its zero set $u^{-1}(0) = \{a_1, \ldots, a_k\}$ (counted with multiplicity, according to the local degree of u around any zero) and orientation. Letting $\mathcal{F}$ be the moduli space of all minimizers, the following quantitative stability estimates hold:*

$$\inf_{(u_0,\nabla_0)\in\mathcal{F}}(\|u - u_0\|^2_{L^2(\mathbb{R}^2)} + \|F_\nabla - F_{\nabla_0}\|^2_{L^2(\mathbb{R}^2)}) \leq C_k(E_1(u, \nabla) - 2\pi k), \tag{13.3}$$

for some constant $C_k > 0$ and all pairs such that the discrepancy $E_1(u, \nabla) - 2\pi k \leqslant \delta_k$ is small enough.

Proof. Existence and uniqueness were proved in [74, 75], while quantitative stability was obtained in [45]. □

The proof of the above theorem uses weighted estimates developed in [44]. Essentially, Theorem 13.4.1 tells us that in the vanishing ε limit, two-dimensional slices perpendicular to the energy concentration set resemble minimizing vortex solutions in $\mathbb{R}^2$. In the case of regular enough pairs (u, ∇), we also have the stability of the Jacobian and the energy density, given by the following theorem.

Theorem 13.4.2. *For any $\Lambda > 1$ and integer k, there exist constants $C_{\Lambda,k} > 0$ and $\eta_{\Lambda,k} > 0$ with the following property. Let $(u, \nabla) \in W^{1,2}_{loc}(\mathbb{R}^2)$ be a finite-energy pair such that*

(i) $\Lambda^{-1}|u_0| \leq |u| \leq \Lambda|u_0|$ *for a solution (u_0, ∇_0) with zero set $\{x_j\}_{j=1}^k$ (counted with multiplicity);*

(ii) $E_1(u, \nabla) - 2\pi k \leq \eta^2_{\Lambda,k}$;

(iii) $\frac{u}{|u|} \in W^{1,1}_{loc}$ *and has the same degree as $\frac{u_0}{|u_0|}$ around each x_j.*

Then for any $0 < \gamma < \frac{1}{k}$, *writing* $\nabla = d - i\alpha$, *we have*

$$\int_{\mathbb{R}^2} |u_0|^{2+2\gamma} \left[\left| d \log \left(\frac{|u|}{|u_0|} \right) \right|^2 + |\alpha - \alpha_0|^2 \right] \leq \frac{C_{\Lambda,k}}{\gamma^2} [E_1(u, \nabla) - 2\pi k],$$

up to a change of gauge. Moreover, the Jacobian and the energy density satisfy the following estimates:

$$\begin{aligned} &\|J(u, \nabla) - J(u_0, \nabla_0)\|_{L^1(\mathbb{R}^2)} + \|e_1(u, \nabla) - e_1(u_0, \nabla_0)\|_{L^1(\mathbb{R}^2)} \\ &\leq C_{\Lambda,k} \sqrt{E_1(u, \nabla) - 2\pi k}. \end{aligned} \tag{13.4}$$

Proof. For the proof see [45, Theorems 1.2 and 1.3], as well as [45, Section 3.3]. □

Chapter 14

Quantifying flatness and the excess

We assume that $n \geqslant 3$ throughout the rest of the paper, unless otherwise stated.

14.1 Excess definitions

In this section we introduce a way to measure *flatness* of a pair (u, ∇). Inspired by the definition of tilt-excess by De Giorgi [24], we define the *Yang–Mills–Higgs excess* as

$$\begin{aligned} \mathbf{E}(u, \nabla, B_r(x), S) &:= \frac{r^{2-n}}{2\pi} \int_{B_r(x)} [e_\varepsilon(u, \nabla) - J(u, \nabla) \wedge e_S^*] \\ &= \mu_\varepsilon(B_r(x)) - \langle \Gamma_\varepsilon, \mathbf{1}_{B_r(x)} e_S^* \rangle \end{aligned} \tag{14.1}$$

for any *oriented* $(n-2)$-plane S in $\mathbb{R}^n$ with the associated $(n-2)$-vector e_S and $(n-2)$-covector e_S^*. Take an oriented orthonormal basis of $S = \mathrm{span}\{e_3, \ldots, e_n\}$ and extend it to an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. Then by a completion of squares we see that the excess splits into two terms:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2,$$

where

$$\mathbf{E}_1(u,\nabla,B_r(x),S) := \frac{r^{2-n}}{2\pi}\int_{B_r(x)}\left[\sum_{k=3}^{n}|\nabla_{e_k}u|^2+\varepsilon^2\sum_{(j,k)\neq(1,2)}\omega(e_j,e_k)^2\right] \tag{14.2}$$

and

$$\mathbf{E}_2(u,\nabla,B_r(x),S) := \frac{r^{2-n}}{2\pi}\int_{B_r(x)}\left[|\nabla_{e_1}u+i\nabla_{e_2}u|^2+\left|\varepsilon\omega(e_1,e_2)-\frac{1-|u|^2}{2\varepsilon}\right|^2\right]. \tag{14.3}$$

Note that $\mathbf{E}_1$ quantifies how flat the solution is in the directions tangent to S, while $\mathbf{E}_2$ quantifies the error in the vortex equations on perpendicular slices. Moreover, $\mathbf{E}_1$ does *not* depend on the orientation of S (while $\mathbf{E}$ and $\mathbf{E}_2$ do).

The Yang–Mills–Higgs excess is a key tool in our analysis. For $S := \{0\}\times\mathbb{R}^{n-2}$, with a slight abuse of notation, we define

$$\mathbf{E}_z = \frac{1}{2\pi}\int_{B_1^2\times\{z\}}[e_\varepsilon(u,\nabla)-J(u,\nabla)(e_1,e_2)]$$

for $z\in\mathbb{R}^{n-2}$, and similarly

$$(\mathbf{E}_1)_z := \frac{1}{2\pi}\int_{B_1^2\times\{z\}}\left[\sum_{k=3}^{n}|\nabla_{e_k}u|^2+\varepsilon^2\sum_{(j,k)\neq(1,2)}\omega(e_j,e_k)^2\right],$$

$$(\mathbf{E}_2)_z := \frac{1}{2\pi}\int_{B_1^2\times\{z\}}\left[|\nabla_{e_1}u+i\nabla_{e_2}u|^2+\left|\varepsilon\omega(e_1,e_2)-\frac{1-|u|^2}{2\varepsilon}\right|^2\right].$$

14.2 The tilt-excess decay statement

Parallel to De Giorgi's [24] and Allard's [2] regularity theorems, we aim to prove a *decay of the excess* up to scale ε, compare with [80, Theorem 3.3]. More precisely, our goal is to show Theorem 10.3.2, which is one of the main results of the present work. For convenience, we recall its statement here.

Theorem 14.2.1. *For any $n \geq 3$ and small enough $0 < \rho \leqslant \rho_0(n)$ there exist constants $C(n) > 0$ and $\varepsilon_0(n,\rho), \tau_0(n,\rho)$ such that the following holds. Let (u,∇) be a critical point for the energy E_ε, given by (12.1), with $\varepsilon \leq \varepsilon_0$. Assume that u satisfies the bounds (13.1) and (13.2), that $u(0) = 0$, and the energy bound*

$$\frac{1}{|B_1^{n-2}|}\int_{B_1^n} e_\varepsilon(u,\nabla) \leq 2\pi + \tau_0.$$

Then at least one of the following statements is true: either

$$\boldsymbol{E}_1(u,\nabla,B_\rho^n,\overline{S}) \leq C(n)\rho^2 \boldsymbol{E}_1(u,\nabla,B_1^n,S), \tag{14.1}$$

for some $(n-2)$-plane $\overline{S}$ with $\|P_{\overline{S}} - P_S\| \leq C(n)\sqrt{\boldsymbol{E}_1(u,\nabla,B_1^n,S)}$, where P_S is the orthogonal projection onto S, the plane minimizing $\boldsymbol{E}(u,\nabla,B_\rho^n,\cdot)$, and $\|\cdot\|$ is the Hilbert–Schmidt norm, or

$$\boldsymbol{E}_1(u,\nabla,B_1^n,S) \leq \max\{C(n)\varepsilon^2|\log \boldsymbol{E}|^2\sqrt{\boldsymbol{E}}, e^{-K(n)/\varepsilon}\}, \tag{14.2}$$

where $\boldsymbol{E} = \boldsymbol{E}(u,\nabla,B_1^n,S)$.

Note that thanks to Proposition 13.2.1 and Proposition 13.2.2, if u is an entire solution such that $\int_{B_R^n} e_\varepsilon(u,\nabla) = O(R^{n-2})$, then (13.1) and (13.2) are statisfied. In particular by scaling we deduce the following.

Theorem 14.2.2. *For any small enough $0 < \rho \leqslant \rho_0(n)$, there exist constants $C(n), R_0(n,\rho) > 0$ and $\tau_0(n,\rho)$ with the following property. Let (u,∇) be an entire critical point for E_1, with the energy bound*

$$\lim_{R\to\infty}\frac{1}{|B_R^{n-2}|}\int_{B_R^n} e_1(u,\nabla) \leq 2\pi + \tau_0.$$

Then for all $R \geqslant R_0$ at least one of the following statements is true: either

$$\boldsymbol{E}_1(u, \nabla, B^n_{\rho R}, \overline{S}) \leq C(n)\rho^2 \boldsymbol{E}_1(u, \nabla, B^n_R, S), \tag{14.3}$$

for some $(n-2)$-plane $\overline{S}$ with $\|P_{\overline{S}} - P_S\| \leq C(n)\sqrt{\boldsymbol{E}_1(u, \nabla, B^n_R, S)}$ and S minimizing $\boldsymbol{E}(u, \nabla, B^n_R, \cdot)$, or

$$\boldsymbol{E}_1(u, \nabla, B^n_R, S) \leq \max\{C(n)R^{-2}|\log \boldsymbol{E}|^2\sqrt{\boldsymbol{E}}, e^{-K(n)R}\}, \tag{14.4}$$

where $\boldsymbol{E} = \boldsymbol{E}(u, \nabla, B^n_R, S)$.

14.3 Blow-up at multiplicity one points

Allard's regularity theorem [2] asserts that the energy concentration set in Theorem 13.1.1 is locally a $C^{1,\alpha}$ submanifold around points of multiplicity one. We use this to show that, for any blow-down, the energy concentration set is a flat $(n-2)$-plane.

Proposition 14.3.1 (Multiplicity one and vanishing of excess)**.** *For any $\delta > 0$ there exist $\tau_0(n, \delta) > 0$ and $\varepsilon_0(n, \delta) > 0$ small enough with the following property. Let (u, ∇) be a critical point for E_ε on the unit ball B^n_1, with $u(0) = 0$ and $\varepsilon \leq \varepsilon_0$, as well as the energy bound*

$$\frac{1}{|B^{n-2}_1|}\int_{B^n_1} e_\varepsilon(u, \nabla) \leq 2\pi + \tau_0$$

and (13.1)–(13.2). Then, after a suitable rotation and, possibly, a conjugation of (u, ∇),

$$\boldsymbol{E}(u, \nabla, B^n_{1/2}, \mathbb{R}^{n-2}) \leq \delta,$$

where we write $\mathbb{R}^{n-2}$ to mean $\{0\} \times \mathbb{R}^{n-2}$. As a consequence, given an entire critical point $(\tilde{u}, \tilde{\nabla})$ for

E_1, with $u(0) = 0$ and the energy bound

$$\lim_{R\to\infty} \frac{1}{|B_R^{n-2}|} \int_{B_R^n} e_1(\tilde{u}, \tilde{\nabla}) \le 2\pi + \tau_0(n),$$

then the previous limit is 2π and we can find oriented $(n-2)$-planes $S(R)$ such that

$$\lim_{R\to\infty} \boldsymbol{E}(\tilde{u}, \tilde{\nabla}, B_R^n, S(R)) = 0.$$

Proof. The proof is a standard argument by compactness and contradiction.

$\boxed{\textit{Local case.}}$ Assume that there are sequences $(u_\varepsilon, \nabla_\varepsilon)$ and $\tau_\varepsilon \to 0$ (as $\varepsilon \to 0$) such that

$$\int_{B_1^n} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \le (2\pi + \tau_\varepsilon)|B_1^{n-2}|$$

and, on the other hand,

$$\liminf_{\varepsilon\to 0} \mathbf{E}(u_\varepsilon, \nabla_\varepsilon, B_{1/2}^n, S(\varepsilon)) > 0, \tag{14.1}$$

for any choice of oriented $(n-2)$-planes $S(\varepsilon)$ (where, with abuse of notation, we write ε to mean a sequence $\varepsilon_k \to 0$). We apply Theorem 13.1.1: up to extracting a subsequence, we have

$$e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)\, dx \overset{*}{\rightharpoonup} 2\pi\, d\mu_V$$

in duality with C_c^0, where V is a stationary integral $(n-2)$-varifold whose weight μ_V obeys the bound

$$\mu_V(B_1^n) \le \liminf_{\varepsilon\to 0} \frac{1}{2\pi} \int_{B_1^n} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \le |B_1^{n-2}|. \tag{14.2}$$

Moreover there exists an integral $(n-2)$-cycle Γ such that $J(u_\varepsilon, \nabla_\varepsilon) \rightharpoonup 2\pi\Gamma$ as currents and

$|\Gamma| \leq \mu_V$.

Since $u_\varepsilon(0) = 0$, by the clearing-out lemma [62, Corollary 4.4] we get that $0 \in \operatorname{spt}(\mu_V)$, so that $\Theta^{n-2}(\mu_V, 0) \geq 1$ since V is an integral stationary varifold. Because of (14.2), the monotonicity formula for stationary varifolds is saturated, showing that V must be a cone with respect to the origin; we extend it to a stationary cone $\tilde{V}$ on $\mathbb{R}^n$. Since $\Theta^{n-2}(\mu_{\tilde{V}}, x) \geqslant 1 = \Theta^{n-2}(\mu_{\tilde{V}}, 0)$ for all $x \in \operatorname{spt}(\mu_{\tilde{V}})$, we see that $\tilde{V}$ is a cone with respect to any $x \in \operatorname{spt}(\mu_{\tilde{V}})$, and hence a plane (since the tangent plane exists for a.e. point x). Thus, up to a rotation, V is the multiplicity-one varifold associated to $\{0\} \times \mathbb{R}^{n-2}$.

Moreover, the argument used in [62, Section 6.2] to show integrality of V actually reveals that the limiting density is the sum of the absolute values of the degrees of

$$\frac{u_\varepsilon}{|u_\varepsilon|}\Big|_{\partial D_i \times \{z\}}$$

along typical slices $B_1^2 \times \{z\}$ with $|z| < \frac{1}{2}$, where $D_1, \dots, D_N \subset B_{1/2}^2$ are suitable disjoint disks (depending on z) such that $u_\varepsilon(\cdot, z) \neq 0$ on $B_{1/2}^2 \setminus \bigcup_i D_i$ (see in particular the proof of [62, Proposition 6.6] and the conclusion of [62, Proposition 6.7]). Since the limiting density is 1 and, eventually, $u_\varepsilon(y, z) \neq 0$ for $y \in \partial B_{1/2}^2$ and $z \in B_{1/2}^{n-2}$, we see that

$$\deg \frac{u_\varepsilon}{|u_\varepsilon|}(\cdot, z) = 1 \quad \text{from } \partial B_{1/2}^2 \text{ to } S^1$$

eventually. As in [62, Lemma 6.11], we deduce that $\Gamma = \pm [\![\{0\} \times B_1^{n-2}]\!]$. We conclude that, after possibly replacing (u, ∇) with the conjugate pair, we have

$$
\begin{aligned}
0 &< \lim_{\varepsilon \to 0} \mathbf{E}(u_\varepsilon, \nabla_\varepsilon, B^n_{1/2}, \mathbb{R}^{n-2}) \\
&= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{B^n_1} [e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) - J(u_\varepsilon, \nabla_\varepsilon) \wedge e_3^* \wedge \cdots \wedge e_n^*] \\
&= \mu_V(B^n_{1/2}) - \langle \Gamma, \mathbf{1}_{B^n_{1/2}} e_3^* \wedge \cdots \wedge e_n^* \rangle \\
&= 0,
\end{aligned}
$$

which is the desired contradiction.

Entire case. For the case of an entire solution (u, ∇), we perform a rescaling: writing $\nabla = d - i\tilde{\alpha}$, let

$$
u_\varepsilon(x) := u(\varepsilon^{-1}x), \quad \nabla_\varepsilon := d - i\alpha_\varepsilon \text{ with } \alpha_\varepsilon(x) := \varepsilon^{-1}\alpha(\varepsilon^{-1}x).
$$

Again, by applying Theorem 13.1.1, up to extracting a subsequence we have $e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)\, dx \overset{*}{\rightharpoonup} d\mu_V$, for a stationary integral $(n-2)$-varifold V, and the Jacobians $J(u_\varepsilon, \nabla_\varepsilon) \rightharpoonup \Gamma$ for an integral $(n-2)$-cycle Γ with the pointwise bound $|\Gamma| \leq \mu_V$. Using the monotonicity formula for E_ε, we see that

$$
\mu_V(B^n_R) = \lim_{\varepsilon \to 0} \frac{\varepsilon^{n-2}}{2\pi} \int_{B^n_{R/\varepsilon}} e_1(u, \nabla)
$$

is a constant multiple of R^{n-2}, and hence V is a cone around the origin with $\mu_V(B^n_1) \leq 1 + \frac{\tau_0}{2\pi}$. Then, by Allard's regularity theorem [2], we see that for $\tau_0(n)$ small enough, after a suitable rotation, V is the varifold associated to $\{0\} \times \mathbb{R}^{n-2}$. In particular, this shows the conclusion on the energy limit.

As before, we also have $\Gamma = \pm[\![\{0\} \times \mathbb{R}^{n-2}]\!]$, concluding that for $R := \varepsilon_k^{-1}$ (where $\varepsilon_k \to 0$ is our subsequence) the statement holds for the plane $S(R) := \{0\} \times \mathbb{R}^{n-2}$, either for (u, ∇) or the conjugate pair $(\overline{u}, \overline{\nabla} = d + i\alpha)$ (depending on R). Since the initial sequence $\varepsilon_k \to 0$ was arbitrary, we deduce that

$$\min_{S\in \mathrm{Gr}(n,n-2)} \min\{\mathbf{E}(u,\nabla,B^n_R,S),\mathbf{E}(\overline{u},\overline{\nabla},B^n_R,S)\}\to 0 \tag{14.3}$$

as $R\to\infty$. Finally, letting $S(R)$ realize the minimum over $S\in \mathrm{Gr}(n,n-2)$, since $\mathbf{E}_1\leqslant \mathbf{E}$ does not distinguish between (u,∇) and $(\overline{u},\overline{\nabla})$ we have

$$\mathbf{E}_1(u,\nabla,B^n_R,S(R))\to 0.$$

As a consequence, we must have

$$\sup_{R'\in[R,2R]} \|P_{S(R')}-P_{S(R)}\|\to 0, \tag{14.4}$$

since otherwise we would find sequences $S(R_k)\to\hat{S}$ and $S(R_k')\to\hat{S}'\neq\hat{S}$ (with $R_k\leqslant R_k'\leqslant 2R_k$) for which

$$\mathbf{E}_1(u,\nabla,B^n_R,\hat{S})+\mathbf{E}_1(u,\nabla,B^n_R,\hat{S}')\to 0 \quad \text{as } R=R_k\to\infty,$$

thanks to the assumption $\int_{B^n_R} e_1(u,\nabla)=O(R^{n-2})$. If $\hat{S}\cup\hat{S}'$ spans $\mathbb{R}^n$, this immediately gives $\int_{B^n_R} e_1(u,\nabla)=o(R^{n-2})$, contradicting the assumption $u(0)=0$ and the clearing-out lemma. Otherwise, their span is $(n-1)$-dimensional; letting e_1 be a unit vector orthogonal to it and completing to an orthonormal basis $\{e_1,\dots,e_n\}$ such that $e_2\perp\hat{S}$, we deduce that

$$\int_{B^n_R}\left[\sum_{j=2}^n |\nabla_{e_j}u|^2+|\omega|^2\right]=o(R^{n-2}).$$

Because of (14.3), we also have

$$\min\{\mathbf{E}_2(u,\nabla,B^n_R,\hat{S}),\mathbf{E}_2(\overline{u},\overline{\nabla},B^n_R,\hat{S})\}\to 0,$$

giving

$$\int_{B_R^n}\left[||\nabla_{e_1}u|-|\nabla_{e_2}u||^2+\left||\omega(e_1,e_2)|-\frac{1-|u|^2}{2}\right|^2\right]\to 0,$$

giving again the contradiction $\int_{B_R^n} e_1(u,\nabla)=o(R^{n-2})$.

Having established (14.4), the claim follows by a straightforward continuity argument: for R large enough we cannot have that

$$\mathbf{E}(u,\nabla,B_R^n,S(R)),\quad \mathbf{E}(\overline{u},\overline{\nabla},B_{R'}^n,S(R'))$$

are both small, for some $R'\in[R,2R]$, since this would imply that

$$\mathbf{E}_2(u,\nabla,B_R^n,S(R))+\mathbf{E}_2(\overline{u},\overline{\nabla},B_R^n,S(R))$$

is also small, which would give again small normalized energy on B_R^n; the same holds interchanging the roles of R and R', completing the proof. □

We also record the following consequence of the Hausdorff convergence of the vorticity set $Z=\{|u|\le\frac{3}{4}\}$.

Lemma 14.3.2 (Soft height bound)**.** *For any $\sigma>0$ there exist $\tau_0(n,\sigma)>0$ and $\varepsilon_0(n,\sigma)>0$ with the following property. Let (u,∇) be a critical point for E_ε on B_1^n, with $\varepsilon\le\varepsilon_0$ and $u(0)=0$, as well as the energy bound*

$$\frac{1}{|B_1^{n-2}|}\int_{B_1^n} e_\varepsilon(u,\nabla)\le 2\pi+\tau_0$$

and (13.1)–(13.2). Then, after a suitable rotation, the zero set is contained in a small neighborhood of $\mathbb{R}^{n-2}$; more precisely,

$$\{|u_\varepsilon|\le 3/4\}\cap B_{1-\sigma}^n\subset B_\sigma^2\times B_1^{n-2}.$$

Proof. Following the same strategy as in the proof of Proposition 14.3.1, the statement follows from the Hausdorff convergence of the vorticity set in Theorem 13.1.1. □

Remark 14.3.3. We will often use the following observation: if the excess $\mathbf{E}_1$ is suitably small on B_1^n, then the same conclusion holds without any rotation. The same holds under other assumptions forcing the vorticity set to concentrate on the plane $\mathbb{R}^{n-2}$ in the limit $\varepsilon \to 0$, such as energy close to $|B_1^{n-2}| \cdot 2\pi$ on the cylinder $B_1^2 \times B_1^{n-2}$, for a critical pair defined there (with $u(0) = 0$).

In the following lemma we essentially show that if $\mathbf{E}_1$ is small in a ball of radius larger than ε, then $\mathbf{E}$ is small as well.

Lemma 14.3.4 **($\mathbf{E}_1$ vanishing implies $\mathbf{E}$ vanishing).** *For any $\delta, \Lambda > 0$ there exist $\tau_0(n, \delta, \Lambda) > 0$ and $\varepsilon_0(n, \delta, \Lambda) > 0$ small enough with the following property. Let (u, ∇) be a critical pair for E_ε on the unit ball B_1^n, with $u(0) = 0$,*

$$E_\varepsilon(u, \nabla) \leqslant 2\pi + \tau_0,$$

and (13.1)–(13.2), as well as $\varepsilon \leqslant \varepsilon_0$. Let $x \in B_{1-\delta}^n$ be a point such that

$$\sup_{\varepsilon \leqslant s \leq 1-|x|} \boldsymbol{E}_1(u, \nabla, B_s^n(x), \mathbb{R}^{n-2}) \leq \tau_0.$$

Then, up to conjugating the pair,

$$\sup_{\varepsilon \leqslant s \leq 1-|x|} \boldsymbol{E}(u, \nabla, B_{s/2}^n(x), \mathbb{R}^{n-2}) \leq \delta.$$

Proof. The proof of this lemma is basically the equivalence of the (second-order) Euler–Lagrange equations and the (first-order) vortex equations in two dimensions.

By contradiction, assume we have a sequence (u_k, ∇_k) of critical points for E_ε, with $\varepsilon = \varepsilon_k \to 0$, and a sequence of points $x_k \in B^n_{1-\delta}$ and radii $s_k \in [\varepsilon_k, 1-|x_k|]$ such that

$$\mathbf{E}_1(u_k, \nabla_k, B^n_{s_k}(x_k), \mathbb{R}^{n-2}) \to 0, \quad \liminf_{k\to\infty} \mathbf{E}(u_k, \nabla_k, B^n_{s_k/2}(x_k), \mathbb{R}^{n-2}) \geq \delta.$$

We now distinguish a few cases depending on the behavior of the limit ε_k/s_k, which we can assume to exists and to belong to $[0,1]$, and on the distance of x_k from the vorticity set $Z_k = \{x \in B^n_{s_k}(x_k) : |u_k| \leqslant 3/4\}$.

$\boxed{\textit{Case 1: } \varepsilon_k/s_k \to 0 \textit{ and } \operatorname{dist}(x_k, Z_k)/s_k \to 0.}$ Since the energy concentration varifold is a plane with multiplicity 1 (as in the previous proof), recalling that $1-|x_k| \geqslant \delta$ and x_k has vanishing distance from the vorticity set, we immediately see that

$$\frac{1}{|B^{n-2}_{1-|x_k|}|}\int_{B_{1-|x_k|}(x_k)} e_{\varepsilon_k}(u_k, \nabla_k) \to 2\pi.$$

Defining the map $\phi_k(x) := x_k + s_k x$, we consider the pullback pair

$$(\tilde{u}_k, \tilde{\nabla}_k) := \phi_k^*(u_k, \nabla_k),$$

which is critical for $E_{\tilde{\varepsilon}_k}$, where $\tilde{\varepsilon}_k := \varepsilon_k/s_k$. Moreover, we have

$$\limsup_{k\to\infty} \frac{1}{|B^{n-2}_1|}\int_{B^n_1} e_{\tilde{\varepsilon}_k}(\tilde{u}_k, \tilde{\nabla}_k) \leqslant 2\pi$$

by monotonicity of the energy.

Since $\tilde{\varepsilon}_k \to 0$ and 0 has vanishing distance from $\{|\tilde{u}_k| \leqslant \frac{3}{4}\} \cap B^n_1$, as in the previous proof, the energy concentration varifold V is a plane S passing through the origin, with multiplicity 1, while the limiting cycle $\Gamma = \pm[\![S]\!]$. By possibly replacing $(\tilde{u}, \tilde{\nabla})$ with their conjugate, we can assume that $\Gamma = [\![S]\!]$. Also, the stress-energy tensors

$$T_{\tilde{\varepsilon}_k}(\tilde{u}_k, \tilde{\nabla}_k),$$

viewed as matrix-valued measures, converge (up to subsequences) to a limit T such that $dT(x) = P_{T_x V}\, d\mu_V(x)$, where $P_{T_x V}$ is the orthogonal projection onto the tangent space $T_x V$ (cf. [62, Section 6.1]). Hence, the fact that $\mathbf{E}_1(\tilde{u}_k, \tilde{\nabla}_k, B_1^n, \mathbb{R}^{n-2}) \to 0$ implies $T_x V = \mathbb{R}^{n-2}$ a.e., giving $S = \mathbb{R}^{n-2}$. Since

$$\lim_{k\to\infty} \int_{B_{1/2}^n} [e_{\tilde{\varepsilon}_k}(\tilde{u}_k, \tilde{\nabla}_k) - J(\tilde{u}_k, \tilde{\nabla}_k) \wedge e_S^*] = 2\pi[\mu_V(B_{1/2}^n) - \langle \Gamma, \mathbf{1}_{B_{1/2}^n} e_S^* \rangle] = 0$$

, we get the desired contradiction in this case.

$\boxed{\textit{Case 2: } \varepsilon_k/s_k \to 0 \textit{ and } \operatorname{dist}(x_k, Z_k)/s_k \to 2d > 0.}$ By applying the same scaling as in the previous step we get that $|\tilde{u}_k|$ converges uniformly to 1 in $B_d^n(0)$, which immediately implies that both excesses converges to 0 in $B_{ds_k}^n(x_k)$ and thus the statement of the theorem with $s/2$ replaced by ds. A covering argument then allows to pass to $s/2$.

$\boxed{\textit{Case 3: } \varepsilon_k/s_k \to \bar{\varepsilon} > 0.}$ Note that this implies that $s_k \to 0$.

After passing to a local Coulomb gauge, for any $\ell \in \mathbb{N}$ we get local uniform C^ℓ bounds on $B_{R_k}^n$, with $R_k := \delta/s_k$, since by monotonicity we have local uniform bounds on the energy here, see [62, Appendix]. By Arzelá–Ascoli we obtain a subsequential limit $(\tilde{u}_\infty, \tilde{\nabla}_\infty)$ in $C^\infty(\mathbb{R}^n)$. By the definition of $\mathbf{E}_1$ (cf. (14.2)), we see that $(\tilde{\nabla}_\infty)_{\partial_k}\tilde{u}_\infty = 0$ for all $3 \le k \le n$ and $\tilde{\omega}_\infty(e_j, e_k) = 0$ for all $(j,k) \neq (1,2)$. As in [62, Proposition 6.7] (after [62, eq. (6.30)]), up to a further change of gauge, the limiting pair depends only on the first two coordinates. By the equivalence of first-order and second-order vortex equations in $\mathbb{R}^2$ [75] (cf. also the end of the proof of [62, Proposition 6.7]), we see that $(\tilde{u}_\infty, \tilde{\nabla}_\infty)$ solves the first-order vortex equations up to conjugation; this yields a contradiction for k large enough. □

Lemma 14.3.5. *For every $\sigma > 0$ there exist constants $\eta(n,\sigma), C(n,\eta) > 0$ such that if $r \geqslant C\varepsilon$ and*

(u,∇) is a critical pair on $B_r(p)$ satisfying (13.1)–(13.2) and $\boldsymbol{E}_1(u,\nabla,B^n_r(p),\mathbb{R}^{n-2})\leqslant\eta$ then

$$\{|u|\leqslant 3/4\}\cap B^n_{(1-\sigma)r}(p)\subseteq B^2_{\sigma r}(y)\times\mathbb{R}^{n-2},$$

provided that $|u(p)|\leqslant\frac{3}{4}$ at $p=(y,z)$ and the normalized energy is at most $2\pi+\eta$.

Moreover, given $\sigma,\Lambda>0$ there are $\eta(n,\sigma,\Lambda),C(n,\eta,\Lambda)>0$ such that if

$$\boldsymbol{E}_1(u,\nabla,B^n_{C\varepsilon}(p),\mathbb{R}^{n-2})\leqslant\eta$$

then $G:=\{u=0\}\cap B^n_{\Lambda\varepsilon}(p)$ is a σ-Lipschitz graph, we have the inclusion

$$\{|u|\leqslant 3/4\}\cap B^n_{\Lambda\varepsilon}(p)\subseteq B_{C(n)\varepsilon}(G),$$

and $\varepsilon|u|$ is comparable with the distance from S in this neighborhood.

Proof. The first part follows by the very same arguments of Lemma 14.3.4. The second one is again showed by contradiction after scaling by ε, noticing that in the Coulomb gauge the contradicting sequence (u_k,∇_k) converges smoothly to a solution depending only on the two variables (y_1,y_2). To infer the smooth convergence of the zero set (which is gauge invariant) one notices that, by the explicit form of the Taubes solution, the Jacobian $Ju_k(e_1,e_2)$ is bounded away from zero. Convergence of the zero set then follows from the implicit function theorem. Compare also with the proof of Proposition 15.3.1. □

In the next lemma we show that the energy on each slice is approximately the excess on the slice plus the degree of u on the boundary.

Lemma 14.3.6. *Let (u,∇) be an arbitrary smooth pair defined on $\overline{B}^2_1\times\overline{B}^{n-2}_1$ (not necessarily a* *critical point) with*

$$e_\varepsilon(u,\nabla)(x) \le e^{-K/\varepsilon} \quad \textit{for all } x \in \partial B_1^2 \times B_1^{n-2}$$

and $|u(x)| \geqslant \frac{1}{2}$ *for all* x *in the same set. Then we have*

$$\left| \deg(u/|u|, \partial B_1^2 \times \{z\}) + \boldsymbol{E}_z - \frac{1}{2\pi}\int_{B_1^2\times\{z\}} e_\varepsilon(u,\nabla) \right| \le 4\varepsilon e^{-K/\varepsilon},$$

for all $z \in B_1^{n-2}$*, up to conjugating the pair.*

Proof. First of all, by a completion of squares, since

$$J := J(u,\nabla)(e_1,e_2) = 2\langle i\nabla_1 u, \nabla_2 u\rangle + (1-|u|^2)\omega(e_1,e_2),$$

we see that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{B_1^2\times\{z\}} e_\varepsilon(u,\nabla) \\
&= (\mathbf{E}_1)_z + \frac{1}{2\pi}\int_{B_1^2\times\{z\}} \left[|\nabla_1 u|^2 + |\nabla_2 u|^2 + \varepsilon^2\omega(e_1,e_2)^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2} \right] \\
&= (\mathbf{E}_1)_z + \frac{1}{2\pi}\int_{B_1^2\times x} \left[|i\nabla_1 u - \nabla_2 u|^2 + \left|\varepsilon\omega(e_1,e_2) - \frac{1-|u|^2}{2\varepsilon}\right|^2 + J \right] \\
&= \mathbf{E}_z + \frac{1}{2\pi}\int_{B_1^2\times\{z\}} J.
\end{aligned} \tag{14.5}$$

We then define the modulus $r : B_1^2 \to [0,\infty)$ and the phase $\theta : B_1^2 \setminus \{r=0\} \to S^1$ by

$$r(y) := |u(y,z)|, \quad \theta(y) := \frac{u}{|u|}(y,z).$$

Writing $\nabla = d - i\alpha$, we also have

$$r^2(d\theta - \alpha)(y) = \langle \nabla u, iu\rangle(y,z)$$

(note that θ and α are not gauge-invariant). Recalling that $J(u,\nabla) = d\alpha + d\langle \nabla u, iu\rangle$, we compute

$$\int_{B_1^2\times\{z\}} J(u,\nabla) = \int_{\partial B_1^2\times\{z\}} [(1-r^2)\alpha(\tau) + r^2\partial_\tau\theta],$$

where τ is the tangent vector to ∂B_1^2. Hence, we have

$$\int_{B_1^2\times\{z\}} J(u,\nabla) = 2\pi\deg(u/|u|, \partial B_1^2\times\{z\}) + \int_{\partial B_1^2\times\{z\}} (1-r^2)[\alpha(\tau) - \partial_\tau\theta],$$

and the last integrand is bounded by

$$(|u|^{-2} - 1)|\langle \nabla u, iu\rangle| \leqslant 4(1-|u|^2)|\nabla u| \leqslant 4\varepsilon e_\varepsilon(u,\nabla)$$

in absolute value. Combining these bounds, the claim follows. □

Chapter 15

Slicing the current and Lipschitz approximation

In this section, inspired by [5, 25, 53], we slice the currents Γ_ε dual to the Jacobians $J(u, \nabla)$. We get metric-space-valued functions of bounded variation (MBV) in the sense of Ambrosio [3], with values in 0-currents in $\mathbb{R}^2$. Then, by placing a threshold on the maximal function of $\mathbf{E}_1$, we construct a Lipschitz approximation of the barycenter of each slice with a uniform $W^{1,2}$ bound.

15.1 Slicing identities and BV estimates

We start by defining vertical slices.

Definition 15.1.1. We define the *vertical slices* of the current Γ_ε, $(\Gamma_\varepsilon)_z = \langle \Gamma_\varepsilon, P, z\rangle$, by the following identity:

$$\int_{B_1^{n-2}(0)} \langle (\Gamma_\varepsilon)_z, \psi\rangle \phi(z)\, dz = \langle \Gamma_\varepsilon, \psi(y)\phi(z) dz\rangle,$$

for any two functions $\psi \in C_c^\infty(B_1^2)$ and $\phi \in C_c^\infty(B_1^{n-2})$, where $P : \mathbb{R}^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ is the projection on the last $n-2$ coordinates.

In the next lemma we derive BV estimates for the slices, given a smooth pair (u, ∇) defined on $B_1^2 \times B_1^{n-2}$.

Lemma 15.1.2 (BV-type estimate)**.** *Define the function $\Phi_\psi : B_1^{n-2} \to \mathbb{R}$ by*

$$\Phi_\psi(z) := \langle (\Gamma_\varepsilon)_z, \psi \rangle.$$

Then, assuming $\int_{B_1^2 \times B_1^{n-2}} e_\varepsilon(u, \nabla) \leqslant 2\pi\Lambda$, the total variation of $\Phi_\psi(x)$ is bounded by $\boldsymbol{E}_1$ and $\boldsymbol{E}$ as follows:

$$\frac{1}{2}|D\Phi_\psi|(B_1^{n-2})^2 \leq \|d\psi\|_{L^\infty}^2 \Lambda \min\{C(n)\boldsymbol{E}_1, \boldsymbol{E}\},$$

where $|D\Phi_\psi|$ denotes the total variation measure, and $\boldsymbol{E}$ and $\boldsymbol{E}_1$ are measured on $B_1^2 \times B_1^{n-2}$ (without normalization).

Proof. The notation and line of argument is inspired from [25, Lemma A.1]. For any $\phi \in C_c^\infty(B_1^{n-2}, \mathbb{R}^{n-2})$ we define the $(n-3)$-form α by

$$\alpha := \sum_{k=3}^{n} (-1)^{k-1} \phi_k(x_3, \ldots, x_n) dx_3 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \cdots \wedge dx_n,$$

so that

$$d\alpha = (\operatorname{div} \phi)(z) dz.$$

Now, writing $x = (y, z)$, we have

$$\begin{aligned}\int_{B_1^{n-2}} \Phi_\psi(z) \div \phi(z)\, dz &= \int_{B_1^{n-2}} \langle (\Gamma_\varepsilon)_z, \psi\rangle (\div\phi)(z)\, dz \\ &= \langle \Gamma_\varepsilon, \psi(y)(\div\phi)(z) dz\rangle \\ &= \langle \Gamma_\varepsilon, d(\psi\alpha)\rangle - \langle \Gamma_\varepsilon, d\psi \wedge \alpha\rangle \\ &= -\langle \Gamma_\varepsilon, d\psi \wedge \alpha\rangle,\end{aligned}$$

where the last equality follows from the fact that $\partial\Gamma_\varepsilon = 0$. Now notice that $d\psi \wedge \alpha$ is a linear combination of $(n-2)$-covectors of the form

$$dx_j \wedge dx_3 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \cdots \wedge dx_n \quad \text{with } j = 1, 2,\ k = 3, \ldots, n.$$

As a consequence,

$$\begin{aligned}&|\langle \Gamma_\varepsilon, d\psi \wedge \alpha\rangle| \\ &\leq \|d\psi\|_{L^\infty} \|\alpha\|_{L^\infty} \sum_{\substack{j=1,2 \\ k=3,\ldots,n}} \int_{B_1^2 \times B_1^{n-2}} [2|\langle i\nabla_{e_j} u, \nabla_{e_k} u\rangle| + (1-|u|^2)|\omega(e_j, e_k)|],\end{aligned}$$

which, by Cauchy–Schwarz, is bounded by

$$\|d\psi\|_{L^\infty} \|\alpha\|_{L^\infty} \cdot C(n)\sqrt{\Lambda}\sqrt{\mathbf{E}_1}.$$

Taking the supremum over the functions ϕ with $\|\phi\|_{L^\infty} \leq 1$, we get the BV bound

$$|D\Phi_\psi|(B_1^{n-2}) \leq C(n)\|d\psi\|_{L^\infty}\sqrt{\Lambda}\sqrt{\mathbf{E}_1}.$$

We can also estimate in the following way. Set $B := B_1^2 \times B_1^{n-2}$ and

$$\vec{e}_{n-2} = e_3 \wedge \cdots \wedge e_n, \quad e_{n-2}^* := dx_3 \wedge \cdots \wedge dx_n,$$

and let us write $d\Gamma_\varepsilon = \overrightarrow{\Gamma}_\varepsilon \, d|\Gamma_\varepsilon|$ (viewing Γ_ε as a measure with values in $\Lambda_{n-2}\mathbb{R}^n$). Since $d\psi \wedge \alpha$ does not have any e_{n-2}^*-component, if we write $\vec{\Gamma}_\varepsilon = (\vec{\Gamma}_\varepsilon \cdot \vec{e}_{n-2})\vec{e}_{n-2} + \vec{R}$ (where the dot denotes the scalar product in $\Lambda^{n-2}\mathbb{R}^n$), we get

$$\langle \Gamma_\varepsilon, d\psi \wedge \alpha \rangle = \int_B \vec{R} \cdot (d\psi \wedge \alpha) \, d|\Gamma_\varepsilon|,$$

and moreover

$$\begin{aligned} \int_B |\vec{R}|^2 \, d|\Gamma_\varepsilon| &= \int_B (1 - (\vec{\Gamma}_\varepsilon \cdot \vec{e}_{n-2})^2) \, d|\Gamma_\varepsilon| \\ &\leq 2 \int_B (1 - \vec{\Gamma}_\varepsilon \cdot \vec{e}_{n-2}) \, d|\Gamma_\varepsilon| \\ &= 2e(\Gamma_\varepsilon, B, \vec{e}_{n-2}), \end{aligned}$$

where $e(\Gamma_\varepsilon, B, \vec{e}_{n-2})$ is the *current excess* defined by

$$e(\Gamma_\varepsilon, B, \vec{e}_{n-2}) := \frac{1}{2} \int_B |\overrightarrow{\Gamma}_\varepsilon - \vec{e}_{n-2}|^2 \, d|\Gamma_\varepsilon|.$$

Hence,

$$\begin{aligned} |\langle \Gamma_\varepsilon, d\psi \wedge \alpha \rangle| &= \left| \int_B \overrightarrow{R} \cdot (d\psi \wedge \alpha) \, d|\Gamma_\varepsilon| \right| \\ &\leq |d\psi \wedge \alpha| \int_B |\vec{R}| \, d|\Gamma_\varepsilon| \\ &\leq \|d\psi\|_{L^\infty} \|\alpha\|_{L^\infty} \sqrt{2e(\Gamma_\varepsilon, B, \vec{e}_{n-2})} \sqrt{|\Gamma_\varepsilon|(B)} \, . \end{aligned}$$

Again, taking the supremum over the functions ϕ with $\|\phi\|_{L^\infty} \le 1$, we get

$$|D\Phi_\psi|(B_1^{n-2}) \le \|d\psi\|_{L^\infty} \sqrt{2e(\Gamma_\varepsilon, B, \overrightarrow{e}_{n-2})}\sqrt{|\Gamma_\varepsilon|(B)}.$$

From the pointwise bound of the Jacobian $|\Gamma_\varepsilon| \le \frac{1}{2\pi} e_\varepsilon(u, \nabla)$ we see that

$$e(\Gamma_\varepsilon, B, \overrightarrow{e}_{n-2}) = |\Gamma_\varepsilon|(B) - \langle \Gamma_\varepsilon, \mathbf{1}_B \overrightarrow{e}_{n-2}\rangle \le \mathbf{E}.$$

The previous bounds, together with $|\Gamma_\varepsilon|(B) \le \Lambda$, give the conclusion. □

Remark 15.1.3. The Jerrard–Soner-type computations in Lemma 15.1.2 are valid for any current without boundary (formally dual to a closed form). In the case of the Yang–Mills–Higgs Jacobian, we record the following identity for convenience (it will be used in Proposition 15.2.1 and Proposition 16.1.2):

$$\begin{aligned}\langle d\Phi_\psi, \phi\rangle = \frac{1}{2\pi}\int_{B_1^2 \times B_1^{n-2}} \sum_{j=1,2}\sum_{k=3}^{n} (-1)^j [\langle 2i\nabla_{e_j} u, \nabla_{e_k} u\rangle \\ + (1-|u|^2)\omega(e_j, e_k)]\partial_{e_{3-j}}\psi\, \phi_k,\end{aligned} \tag{15.1}$$

for any $\phi \in C_c^1(B_1^{n-2}, \mathbb{R}^2)$.

15.2 Lipschitz approximation of the barycenter

Parallel to the regularity theory of minimal currents, we define a Lipschitz approximation of the barycenter of the slices of Γ_ε (see for instance [25, Lemma A.2]). First we fix some notation which will be used frequently:

- we use $\mathbf{E}_1$ as shorthand for $\mathbf{E}_1(u, \nabla, B_1^2 \times B_1^{n-2}, \mathbb{R}^{n-2})$, and similarly for $\mathbf{E}$;
- as already mentioned, for any $z \in B_1^{n-2}$ we denote the excess on the slice $B_1^2 \times \{z\}$ by $(\mathbf{E}_1)_z$,

and similarly for $\mathbf{E}_z$;

- we write $M_{\mathbf{E}_1}(z)$ to denote the maximal function of $(\mathbf{E}_1)_z$;
- we fix a cut-off function $\chi \in C_c^\infty(B^2_{3/4})$ such that $0 \leqslant \chi \leqslant 1$ and $\chi = 1$ on $B^2_{1/2}$.

Proposition 15.2.1 (Lipschitz approximation). *Given $0 < \eta \leqslant \eta_0(n)$ small enough, there exist $\tau_0(n, \eta) > 0$ and $\varepsilon_0(n, \eta) > 0$ such that the following holds. Let (u, ∇) be a critical pair for E_ε, defined on $B^2_1 \times B^{n-2}_1$, satisfying $u(0) = 0$, (13.1)–(13.2), and the energy bound*

$$\frac{1}{|B^{n-2}_1|} \int_{B^2_1 \times B^{n-2}_1} e_\varepsilon(u, \nabla) \leqslant 2\pi + \tau_0.$$

Then, up to conjugating (u, ∇), for $0 < \eta \leqslant \eta_0(n)$ small enough there exists a Lipschitz approximation $h : B^{n-2}_{3/4} \to \mathbb{R}^2$ with the following properties:

(i) $\mathrm{Lip}(h) \leq C\eta$ *and* $\int_{B^{n-2}_{3/4}} |dh|^2 \leq C\boldsymbol{E}_1$;

(ii) $h|_{\mathcal{G}^\eta} = \Phi_{\chi(x_1, x_2)}$ *for a set* $\mathcal{G}^\eta \subseteq B^{n-2}_{3/4}$ *such that* $|B^{n-2}_{3/4} \setminus \mathcal{G}^\eta| \leq C\frac{\boldsymbol{E}_1}{\eta^2}$;

(iii) $\int_{B^2_{3/4} \times (B^{n-2}_{3/4} \setminus \mathcal{G}^\eta)} e_\varepsilon(u, \nabla) \leq C\frac{\boldsymbol{E}_1}{\eta^2} + e^{-K/\varepsilon}$;

(iv) $\int_{\mathcal{G}^\eta} \frac{|dh|^2}{2} \leq (1 + \delta) \int_{\mathcal{G}^\eta} \boldsymbol{E}_z \, dz + e^{-K/\varepsilon}$ *with* $\delta(n, \eta) > 0$ *such that* $\lim_{\eta \to 0} \delta(n, \eta) = 0$.

Here $C = C(n) > 0$ and $K = K(n) > 0$, provided that $\varepsilon \leqslant \varepsilon_0$.

Proof. We define the *good set* to be

$$\mathcal{G}^\eta := \{z \in B^{n-2}_{3/4} \ : \ M_{\mathbf{E}_1}(z) \leq \eta^2\}. \tag{15.1}$$

By the weak L^1 bound and Vitali's covering lemma, we can bound the measure of the complement of the good set, namely the *bad set*, by

$$\mathcal{H}^{n-2}(B^{n-2}_{3/4} \setminus \mathcal{G}^\eta) \leq C(n)\frac{\mathbf{E}_1}{\eta^2}. \tag{15.2}$$

$\boxed{\textit{Bounding energy on the bad set.}}$ To check that the third conclusion holds, we introduce another *bad set* $\mathcal{B}^\eta$, defined on the n-dimensional space: it is the set of points $x = (y, z) \in B^2_{3/4} \times B^{n-2}_{3/4}$ such that, for some radius $r \in (0, \frac{1}{50})$, we have $\mathbf{E}_1(u, \nabla, B^n_r(x), \mathbb{R}^{n-2}) > \eta^2$.

By Vitali's covering lemma, we can cover $\mathcal{B}^\eta$ with balls $B_{5r_i}(x_i)$ such that the balls $B_{r_i}(x_i)$ are disjoint and $\mathbf{E}_1(u, \nabla, B_{r_i}(x_i), \mathbb{R}^{n-2}) > \eta^2$. By monotonicity of the energy, the energy on each dilated ball $B_{5r_i}(x_i)$ is at most $C(n) r_i^{n-2}$, giving

$$\begin{aligned}
\int_{\mathcal{B}^\eta} e_\varepsilon(u, \nabla) &\leqslant \sum_i C(n) r_i^{n-2} \\
&\leqslant \sum_i \frac{C(n)}{\eta^2} r_i^{n-2} \mathbf{E}_1(u, \nabla, B_{r_i}(x_i), \mathbb{R}^{n-2}) \\
&\leqslant \frac{C(n)}{\eta^2} \mathbf{E}_1
\end{aligned}$$

(recall that the excess on a ball $B_r(x)$ is normalized by a factor r^{2-n}). Since the measure of $B^{n-2}_{3/4} \backslash \mathcal{G}^\eta$ obeys the same bound, it is enough to show that

$$\int_S e_\varepsilon(u, \nabla) \leqslant C(n)$$

for η small enough, where $S := (B^2_{3/4} \times \{z\}) \setminus \mathcal{B}^\eta$.

We denote by d_Z the distance from the vorticity set $Z = \{|u| \leqslant \frac{3}{4}\}$. As we can see from the proof of Lemma 14.3.2, its conclusion holds without any rotation in the present situation (as necessarily energy concentrates along $\mathbb{R}^{n-2}$ as $\tau_0, \varepsilon_0 \to 0$). Hence, we can assume that, for any $(y, z) \in S$ on this slice, we have $d_Z(y, z) \geqslant \frac{1}{200}$ unless $|y| < \frac{1}{100}$.

Given $s \geqslant \varepsilon$, by Lemma 14.3.5 we know that if

$$d_Z(y, z), d_Z(y', z) < s,$$

for two points $(y,z),(y',z) \in S$ with $|y|,|y'| < \frac{1}{100}$, then

$$|y - y'| \leqslant Cs,$$

provided that η, ε and ε/s are small enough. With this observation in hand, we can apply Proposition 13.2.3 (giving $e_\varepsilon(u,\nabla)(y,z) \leqslant Ce^{-K\min\{d_Z(y,z),1/10\}/\varepsilon}$ on S) and the coarea formula to write

$$\int_S e_\varepsilon(u,\nabla) \leqslant \frac{C}{\varepsilon^3}\int_0^2 e^{-Kt/\varepsilon}|\{y \in B^2_{1/100} : d_Z(y,z) < t\}|\,dt + \frac{C}{\varepsilon^3}e^{-K/(200\varepsilon)}.$$

The previous observation says that $\{y \in B^2_{1/100} : d_Z(y,z) < t\}$ is included in a ball of radius $C\max\{t,\varepsilon\}$. We deduce that the last integral is bounded by

$$\frac{C}{\varepsilon^3}\int_0^2 e^{-Kt/\varepsilon}C\max\{t^2,\varepsilon^2\}\,dt \leqslant C,$$

giving the desired bound

$$\int_S e_\varepsilon(u,\nabla) \leqslant C(n).$$

$\boxed{\textit{Bounds in terms of } (\boldsymbol{E}_1)_z.}$ Now we establish Dirichlet energy bounds for Φ_ψ on the good set, for $\psi \in C^1_c(B^2_{3/4})$. Given $z \in \mathcal{G}^\eta$, we can use Remark 15.1.3 to bound

$$\begin{aligned}
&|d\Phi_\psi|^2(z)\\
&\leq C\sum_{j=1,2}\sum_{k=3}^{n}\left[\int_{B^2_1\times\{z\}}\Big(|\langle 2i\nabla_{e_j}u,\nabla_{e_k}u\rangle| + (1-|u|^2)|\omega(e_j,e_k)|\Big)\,|\partial_{e_{3-j}}\psi|\right]^2\\
&\leq C\|d\psi\|^2_{L^\infty}\left[\int_{B^2_{3/4}\times\{z\}}|\nabla_{e_1}u|^2 + |\nabla_{e_2}u|^2 + \frac{(1-|u|^2)^2}{\varepsilon^2}\right](\mathbf{E}_1)_z.
\end{aligned}$$

Since $z \in \mathcal{G}^\eta$, we have $S = B^2_{3/4}\times\{z\}$ in the previous argument. Thus, the last integral is bounded by $C(n)$. As a consequence,

$$|d\Phi_\psi|^2(z) \le C(n)\|d\psi\|_{L^\infty}^2(\mathbf{E}_1)_z \quad \text{for all } z \in \mathcal{G}^\eta.$$

$\boxed{\textit{Bounds in terms of } \mathbf{E}_z.}$ Also, we can use Lemma 15.1.2 (cf. [25, Lemma A.2]) to conclude that

$$|d\Phi_{\chi(x_1,x_2)}|^2(z) \leqslant 2\mathbf{E}_z \lim_{r\to 0} \frac{|\Gamma_\varepsilon|(B^2_{1/2} \times B^{n-2}_r(z))}{|B^{n-2}_r|} + Ce^{-K/\varepsilon}.$$

(indeed, this bound follows by applying Lemma 15.1.2 and its proof with $\psi := \chi(ax_1 + bx_2)$ for an arbitrary $(a, b) \in S^1$ and using the fact that this ψ is 1-Lipschitz on $B^2_{1/2}$, outside of which the energy density is exponentially small).

To conclude, we have

$$|\Gamma_\varepsilon|(B^2_{1/2} \times B^{n-2}_r(z)) \leqslant |B^{n-2}_r| + \int_{B^{n-2}_r(z)} \mathbf{E}_z + |B^{n-2}_r|e^{-K/\varepsilon}$$

by Lemma 14.3.6, giving

$$|d\Phi_{\chi(x_1,x_2)}|^2(z) \leqslant 2\mathbf{E}_z(1 + \mathbf{E}_z) + Ce^{-K/\varepsilon},$$

where we can actually replace $\mathbf{E}_z$ with the excess on the smaller disk $B^2_{1/2} \times \{z\}$, denoted by $\mathbf{E}_z(B^2_{1/2})$. Now, fixing $L > 1$ large, by an obvious variant of Lemma 14.3.4 we have that $\mathbf{E}_z(B^2_\varepsilon(y))$ is small for all $y \in B^2_{1/2}$ (see also the remark below). Since we can cover the set $\{y \in B^2_{1/2} : d_Z(y, z) \leqslant L\varepsilon\}$ with $C(n, L)$ such disks, we infer that

$$\begin{aligned}\mathbf{E}_z(B^2_{1/2}) \leqslant \delta(n, L, \eta) + \frac{C(n)}{\varepsilon^3}\int_{L\varepsilon}^2 e^{-Kt/\varepsilon}|\{y \in B^2_{1/2} : d_Z(y, z) < t\}|\,dt \\ + \frac{C(n)}{\varepsilon^3}e^{-K/(4\varepsilon)},\end{aligned}$$

for some quantity $\delta(n, L, \eta)$ vanishing as $\eta \to 0$. Choosing L suitably large, we deduce that

$$\mathbf{E}_z(B^{n-2}_{1/2}) \leqslant \delta(n, \eta) + e^{-K/\varepsilon}$$

for some quantity $\delta(n, \eta)$ vanishing as $\eta \to 0$. The statement follows by extending $h := \Phi_{\chi(x_1,x_2)}|_{\mathcal{G}_\eta}$ to a function with Lipschitz constant $C(n)\eta$. $\square$

Remark 15.2.2. As a technical remark, a simple continuity argument as in Proposition 14.3.1 shows that the possible need of conjugating the pair (u, ∇) in Lemma 14.3.4 happens precisely when the degree of $u/|u|$ along the circle $\partial B^2_{1/2}(0) \times \{0\}$ is -1 instead of 1.

15.3 Lipschitz approximation of the zero set

In this part we collect information about the Lipschitz approximation of the zero set. We use compactness arguments similar to [80, Sectoin 5].

Proposition 15.3.1 (Zero set is Lipschitz on the good set). *For any $\sigma, \delta > 0$, there exists $\eta_0(n, \sigma, \delta)$ small enough with the following property. For (u, ∇) as in the previous statement, for any $\eta \leqslant \eta_0(\delta, \sigma)$, the set $u^{-1}(0) \cap (B^2_{3/4} \times \mathcal{G}^\eta)$ is included in a δ-Lipschitz graph $h_0 : B^{n-2}_{3/4} \to B^2_\sigma$ with the following estimate:*

$$\int_{B^{n-2}_{3/4}} |h_0 - h|^2 \leq C\sigma^2 \frac{\boldsymbol{E}_1}{\eta^2} + C\varepsilon^2 |\log(\boldsymbol{E}_2)|^2 \boldsymbol{E}_2 + Ce^{-K/\varepsilon},$$

for $C = C(n)$ (provided that $\varepsilon \leqslant \varepsilon_0(n, \sigma, \delta)$ and the energy is $\leqslant 2\pi + \tau_0(n, \sigma, \delta)$).

Proof. The proof is similar to [80, Lemma 5.3].

$\boxed{\textit{Lipschitz approximation at scale } \varepsilon.}$ This is essentially the second part of Lemma 14.3.5, but we present a detailed argument here. Notice that, locally at scale ε, critical points enjoy uniform C^k estimates in the Coulomb gauge (and thus C^k bounds for gauge-invariant quantities): see [62, Appendix]. Then around any $x_0 = (y_0, z_0) \in B^2_{3/4} \times \mathcal{G}^\eta$ with $u(x_0) = 0$ we rescale as follows:

$$\tilde{u}(x) := u(x_0 + \varepsilon x), \quad \tilde{\nabla} := \phi^*_{x_0,\varepsilon}(\nabla),$$

where $\phi_{x_0,\varepsilon}$ is the map $x \mapsto x_0 + \varepsilon x$. The resulting pair $(\tilde{u}, \tilde{\nabla})$ satisfies

$$\sup_{r \le 1/(4\varepsilon)} \mathbf{E}_1(\tilde{u}, \tilde{\nabla}, B^2_{1/(4\varepsilon)} \times B^{n-2}_r, \mathbb{R}^{n-2}) \le \eta^2$$

(where the excess is normalized by a factor r^{2-n}). By Arzelá–Ascoli we conclude that, for small enough η, $(\tilde{u}, \tilde{\nabla})$ is C^1-close to a pair (u_0, ∇_0) that satisfies the Yang–Mills–Higgs equations (12.2)–(12.3) and depends only on the variables x_1, x_2 (as in the proof of Lemma 14.3.4). As noted in the proof of Proposition 14.3.1, $u_0/|u_0|$ has degree ± 1 on large circles, and $u(\cdot, z_0)|_{B^2_{3/4}}$

By the main result of [74, 75], we deduce that (u_0, ∇_0) is the standard entire solution of degree ± 1, centered to vanish just at the origin. For this solution, we have

$$|Ju_0|(0) > 0, \tag{15.1}$$

where Ju_0 is the Jacobian of u_0 in the local Coulomb gauge in B^n_1. It then follows that, for small enough $\eta > 0$, we have $|J\tilde{u}(e_1, e_2)| \ge c > 0$. Then, by an application of the implicit function theorem and the fact that $\{\tilde{u} = 0\}$ is a gauge-invariant set, we see that $\{\tilde{u} = 0\}$ is locally a Lipschitz graph with a (qualitatively) small Lipschitz constant. The fact that the zero set intersects the slice only at x_0 follows from Lemma 14.3.5, which says that there is no zero outside a $C\varepsilon$-neighborhood of x_0, while in this neighborhood uniqueness follows from the fact that it holds for u_0 (see also a similar argument in the proof of [45, Theorem 4.1]). Hence, for small enough η, we can define a function $h_0 : \mathcal{G}^\eta \to \mathbb{R}^2$ such that

$$\{u = 0\} \cap (B^2_{3/4} \times \mathcal{G}^\eta) = \mathrm{graph}(h_0).$$

$\boxed{\textit{Lipschitz approximation at larger scales.}}$ By the first part of Lemma 14.3.5, we see that given two points $(y, z), (y', z') \in \{u = 0\} \cap (B^2_{3/4} \times \mathcal{G}^\eta)$ we have

$$|z - z'| \leqslant \delta |y - y'| \quad \text{if } |y - y'| \geqslant C(n, \delta)\varepsilon,$$

for a constant $\delta = \delta(\eta) > 0$ such that

$$\delta(\eta) \to 0 \quad \text{as } \eta \to 0.$$

Together with the previous control at scales comparable with ε, this tells us that h_0 is indeed Lipschitz, with $\mathrm{Lip}(h_0)$ vanishing as $\eta \to 0$. We apply the classical extension theorem to build a Lipschitz extension of h_0 defined on B^{n-2}_s.

$\boxed{L^2 \textit{ estimates.}}$ Using the soft height bound of Lemma 14.3.2 (note that no rotation is needed in the present situation, as necessarily the energy concentrates along $\mathbb{R}^{n-2}$), we have

$$|h| + |h_0| \leqslant \sigma$$

for η (and hence ε) small enough. Using the estimates of Lemma B.0.1 on the good set $\mathcal{G}^\eta$ (see also Remark B.0.3) and the measure bound for the bad set $B^{n-2}_{3/4} \setminus \mathcal{G}^\eta$ we see that

$$\begin{aligned} \int_{B^{n-2}_{3/4}} |h_0 - h|^2 &\leq \int_{\mathcal{G}^\eta} |h_0 - h|^2 + \int_{B^{n-2}_{3/4} \setminus \mathcal{G}^\eta} |h_0 - h|^2 \\ &\leq C\varepsilon^2 |\log \mathbf{E}_2|^2 \mathbf{E}_2 + C\sigma^2 \frac{\mathbf{E}_1}{\eta^2} + Ce^{-K/\varepsilon}. \end{aligned}$$

We thus get the desired conclusion. $\square$

Remark 15.3.2. We remark that the function h_0 is well-behaved under small rotations, since the construction also rotates. However, the Lipschitz approximation of the slice barycenters, a priori, might not behave well under rotations.

Chapter 16

Harmonic approximation and a Caccioppoli-type estimate

16.1 Harmonic approximation

In this section we show that the Lipschitz approximation of Proposition 15.2.1 nearly satisfies the Laplace equation. We achieve this by relating the stress-energy tensor to the slices of Γ_ε using the self-dual discrepancy excess $\mathbf{E}_2$. Then we use this with uniform $W^{1,2}$ bounds to show that the Lipschitz approximation is well approximated in L^2 by a harmonic function. To begin with, we state a very well-known lemma.

Lemma 16.1.1. *For any $\nu > 0$ small there exists $\tau(n, \nu) > 0$ with the following property. Let f be a function in $W^{1,2}(B_1^n)$ such that*

$$\int_{B_1^n} |\nabla f|^2 \leq 1, \quad \left| \int_{B_1^n} \langle df, d\phi \rangle \right| \leq \tau \|d\phi\|_{L^\infty},$$

for any $\phi \in C^1_c(B^n_1)$. Then there exists a harmonic function $w : B^n_1 \to \mathbb{R}$ such that

$$\int_{B^n_1} |dw|^2 \leqslant 1, \quad \int_{B^n_1} |w - f|^2 \leq \nu.$$

Moreover, if f has zero average, we can choose w so that $w(0) = 0$.

Proof. The claim follows easily from Rellich's compact embedding theorem: see for instance [26, Lemma 6.1]. For the second part, by the mean value property of harmonic functions and $\int f = 0$ one gets that

$$|B^n_1||w(0)| = \left|\int_{B_1} w - \int_{B^n_1} f\right| \leqslant C(n)\|w - f\|_{L^2} \leqslant \sqrt{\nu}.$$

The function $w - w(0)$ satisfies the conclusion of the lemma. □

Proposition 16.1.2 (Harmonic approximation)**.** *Let (u, ∇) be a critical point of E_ε as in the previous section and let $h : B^{n-2}_{3/4}$ be the Lipschitz approximation built in Proposition 15.2.1 for η. Then there exist constants $C(n), K(n) > 0$ such that, for any test function $\phi \in C^\infty_c(B^{n-2}_{3/4}, \mathbb{R}^2)$, we have*

$$\left|\int_{B^{n-2}_{3/4}} \langle dh, d\phi\rangle\right| \leq C(\eta^{-1}\boldsymbol{E}_1 + \sqrt{\boldsymbol{E}\boldsymbol{E}_1} + e^{-K/\varepsilon})\|d\phi\|_{L^\infty}.$$

Moreover, given any $\nu > 0$, if $e^{-K/\varepsilon} \leqslant \boldsymbol{E}_1$ and $\boldsymbol{E}$ is small enough (depending on n, η, ν), there exists a harmonic function $w : B^{n-2}_{3/4} \to \mathbb{R}^2$ with $w(0) = 0$ such that

$$\int_{B^{n-2}_{3/4}} |dw|^2 \leq C, \quad \int_{B^{n-2}_{3/4}} |(\boldsymbol{E}_1)^{-1/2}(h - c) - w|^2 \leq \nu,$$

where c is the average of h.

Proof. First, we define the vector field $X := \phi(x_3, \ldots, x_n)e_1$ for any compactly supported test function $\phi \in C^\infty_c(B^{n-2}_{3/4})$, and we test (13.4) with $\psi(x_1, x_2)X$, where ψ is a smooth cut-off function such that $\psi = 1$ on $B^2_{1/2}$ and $\psi = 0$ outside of $B^2_{3/4}$. We obtain

$$\left| \int_{B^2_{1/2} \times B^{n-2}_{3/4}} \langle T_\varepsilon(u, \nabla), DX \rangle \right| \leq C e^{-K/\varepsilon} \|d\phi\|_{L^\infty},$$

thanks to the fact that $d\psi$ is supported in the annulus $B^2_{3/4} \setminus B^2_{1/2}$ and the exponential decay away from the vorticity set Z, which intersects $B^2_{3/4} \times B^{n-2}_{3/4}$ only inside $B^2_{1/4} \times B^{n-2}_{3/4}$. Then, since DX is traceless, we compute

$$\begin{aligned}
&\int_{B^2_{1/2} \times B^{n-2}_{3/4}} \langle T_\varepsilon(u, \nabla), DX \rangle \\
&= -2 \int_{B^2_{1/2} \times B^{n-2}_{3/4}} \sum_{k=3}^{n} \left[\langle \nabla_{e_1} u, \nabla_{e_k} u \rangle + \varepsilon^2 \sum_{j=1}^{n} \omega(e_1, e_j) \omega(e_k, e_j) \right] \partial_{e_k} \phi.
\end{aligned}$$

Except for $j = 2$, the integral of the terms involving the curvature ω is bounded by $C(n)\mathbf{E}_1$, giving

$$\begin{aligned}
&\left| \int_{B^2_{1/2} \times B^{n-2}_{3/4}} \sum_{k=3}^{n} [\langle \nabla_{e_1} u, \nabla_{e_k} u \rangle + \varepsilon^2 \omega(e_1, e_2) \omega(e_k, e_2)] \partial_{e_k} \phi \right| \\
&\leqslant C(\mathbf{E}_1 + e^{-K/\varepsilon}) \|d\phi\|_{L^\infty}.
\end{aligned} \tag{16.1}$$

We now want to relate the expression in the left-hand side with the identity for $d\Phi_{\chi x_1}$ obtained in Remark 15.1.3, which in particular gives

$$\begin{aligned}
&\left| \int_{B^{n-2}_{3/4}} \langle d\Phi_{\chi x_1}, d\phi \rangle \right| \\
&\leqslant C \left| \int_{B^2_{1/2} \times B^{n-2}_{3/4}} \sum_{k=3}^{n} [\langle 2i \nabla_{e_2} u, \nabla_{e_k} u \rangle + (1 - |u|^2) \omega(e_2, e_k)] \partial_{e_k} \phi \right| \\
&\quad + C e^{-K/\varepsilon} \|d\phi\|_{L^\infty},
\end{aligned}$$

or equivalently

$$\begin{aligned}
&\left|\int_{B^{n-2}_{3/4}} \langle d\Phi_{\chi x_1}, d\phi\rangle\right| \\
&\leqslant C\left|\int_{B^2_{1/2}\times B^{n-2}_{3/4}} \sum_{k=3}^{n}\left[-\langle i\nabla_{e_2}u, \nabla_{e_k}u\rangle + \frac{1-|u|^2}{2}\omega(e_k, e_2)\right]\partial_{e_k}\phi\right| \qquad (16.2)\\
&\quad + Ce^{-K/\varepsilon}\|d\phi\|_{L^\infty}.
\end{aligned}$$

We observe that the two integrals in (16.1) and (16.2) differ by the integral of

$$\sum_{k=3}^{n}\left[\langle \nabla_{e_1}u + i\nabla_{e_2}u, \nabla_{e_k}u\rangle + \left(\omega(e_1,e_2) - \frac{1-|u|^2}{2}\right)\omega(e_k,e_2)\right]\partial_{e_k}\phi.$$

Hence, recalling the definition of $\mathbf{E}_2$ and using Cauchy–Schwarz, we conclude that

$$\left|\int_{B^{n-2}_{3/4}} \langle d\Phi_{\chi x_1}, d\phi\rangle\right| \leqslant C(n)(\mathbf{E}_1 + \sqrt{\mathbf{E}_1\mathbf{E}} + e^{-K/\varepsilon})\|d\phi\|_{L^\infty}.$$

Repeating the same for $\Phi_{\chi x_2}$, we arrive at the same conclusion for $\Phi_{\chi(x_1,x_2)}$, integrated against any $\phi \in C^\infty_c(B^{n-2}_{3/4}, \mathbb{R}^2)$. To conclude we note that thanks to items (i) and (ii) of Proposition 15.2.1,

$$\int_{B^{n-2}_{3/4}\setminus \mathcal{G}^\eta} |dh| \leqslant C\eta|B^{n-2}_{3/4}\setminus \mathcal{G}^\eta| \leqslant C\frac{\mathbf{E}_1}{\eta}$$

and, in view of Remark 15.1.3, Cauchy–Schwarz, item (iii) of Proposition 15.2.1 and the assumption $e^{-K/\varepsilon} \leqslant \mathbf{E}_1$,

$$\int_{B^{n-2}_{3/4}\setminus \mathcal{G}^\eta} |d\Phi_{\chi(x_1,x_2)}| \leqslant C\sqrt{\mathbf{E}_1}\left(\int_{B^2_{3/4}\times B^{n-2}_{3/4}\setminus \mathcal{G}^\eta} e_\varepsilon(u,\nabla)\right)^{1/2} \leqslant C\frac{\mathbf{E}_1}{\eta}.$$

The second part follows from Lemma 16.1.1, noting that the normalized function $\tilde{h} := (\mathbf{E}_1)^{-1/2}h$ has Dirichlet energy bounded by $C(n)$ by item (i) of Proposition 15.2.1. □

16.2 Caccioppoli-type estimates

The starting point in the regularity theory of elliptic partial differential equations is the Caccioppoli–Leray bound, obtained by testing the equation with $\phi^2 u$, where ϕ is a cut-off function and u is the solution. We aim to do something similar in spirit. Here the *function* that we deal with is the barycenter of the energy measure at any slice. This suggests that testing the stress-energy tensor with $\phi^2 x_1 e_1$ and $\phi^2 x_2 e_2$ is an appropriate choice.

Proposition 16.2.1. *Let (u, ∇) be a critical point of E_ε as in the previous section. For any $\sigma > 0$ there exist $\varepsilon_0(n, \sigma)$ and $\tau_0(n, \sigma)$ small enough such that the following Caccioppoli-type estimate holds:*

$$\begin{aligned}&\int_{B^{n-2}_{3/4}} \phi^2(z)(\boldsymbol{E}_1)_z \, dz \\ &\leq C \int_{B^2_{1/2}\times B^{n-2}_{3/4}} (x_1^2 + x_2^2) e_\varepsilon(u, \nabla)\Delta\phi^2 + C(\sigma^2 \boldsymbol{E}_1 + e^{-K/\varepsilon})\|D^2\phi\|_\infty,\end{aligned}$$

for any test function $\phi \in C^\infty_c(B^{n-2}_{3/4})$, where $C = C(n)$ and $K = K(n, \sigma)$.

Proof. First, we define the vector fields

$$X := \sum_{k=3}^{n} \partial_k \phi^2(x_3, \dots, x_n)\frac{x_1^2 + x_2^2}{2} e_k, \qquad Y := \phi^2(x_3, \dots, x_n)(x_1 e_1 + x_2 e_2)$$

and we calculate their derivatives:

$$\begin{aligned}DX &= \frac{1}{2}\sum_{3\leq j,k\leq n} \partial^2_{e_j,e_k}\phi^2(x_1^2 + x_2^2) e_j \otimes e_k^* + \sum_{k=3}^{n} \partial_{e_k}\phi^2(x_1 e_k \otimes e_1^* + x_2 e_k \otimes e_2^*) \\ DY &= \phi^2(e_1 \otimes e_1^* + e_2 \otimes e_2^*) + \sum_{k=3}^{n} \partial_{e_k}\phi^2(x_1 e_1 \otimes e_k^* + x_2 e_2 \otimes e_k^*).\end{aligned}$$

Then we test (13.4) with χX and χY, where $\chi = \chi(x_1, x_2)$ is a smooth cut-off function such that $\chi = 1$ on $B^2_{1/2}$ and $\chi = 0$ on $B^2_1 \setminus B^2_{3/4}$. We note that the terms containing $d\chi$ are supported in $(B^2_{3/4} \setminus B^2_{1/2}) \times B^{n-2}_{3/4}$, where $|T_\varepsilon(u,\nabla)| \leqslant C(n)e_\varepsilon(u,\nabla)$ is very small by the exponential decay. Hence,

$$\left| \int_{B^2_{1/2}\times B^{n-2}_{3/4}} \langle T_\varepsilon(u,\nabla), DY\rangle - \langle T_\varepsilon(u,\nabla), DX\rangle \right| \le C\|D^2\phi\|_{L^\infty} e^{-K/\varepsilon}.$$

Using the previous expansion of DX and DY, together with the symmetry of $T_\varepsilon(u,\nabla)$, we see that the above integrand equals

$$\begin{aligned}
&\frac{1}{2}\sum_{3\le j,k\le n} T_\varepsilon(u,\nabla)(e_j,e_k)\partial^2_{e_j,e_k}\phi^2(x_1^2+x_2^2) - \sum_{j=1,2} T_\varepsilon(u,\nabla)(e_j,e_j)\phi^2 \\
&= \frac{x_1^2+x_2^2}{2}\left[e_\varepsilon(u,\nabla)\Delta\phi^2 - 2\sum_{3\leqslant j,k\leqslant n} (\nabla u^*\nabla u + \varepsilon^2\omega^*\omega)(e_j,e_k)\partial^2_{e_j,e_k}\phi^2 \right] \\
&\quad - 2\left[e_\varepsilon(u,\nabla) - |\nabla_{e_1}u|^2 - |\nabla_{e_2}u|^2 - \sum_{j=1,2}\sum_{k=1}^n \varepsilon^2\omega(e_j,e_k)^2 \right]\phi^2.
\end{aligned}$$

By the Modica-type inequality (13.2), the last expression multiplying $-2\phi^2$ is bounded below by

$$\sum_{k=3}^n |\nabla_{e_k}u|^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2} - \varepsilon^2\omega(e_1,e_2)^2 \geqslant \sum_{k=3}^n |\nabla_{e_k}u|^2 + \varepsilon^2 \sum_{(j,k)\neq(1,2)} \omega(e_j,e_k)^2$$

(where the last sum is over all pairs $(j,k) \neq (1,2)$ with $j < k$), which is the integrand in the definition of $\mathbf{E}_1$. Hence, combining the previous bounds, we arrive at

$$\begin{aligned}
&\int_{B^2_{1/2}\times B^{n-2}_{3/4}} \phi^2(z)(\mathbf{E}_1)_z\,dz \\
&\leqslant \int_{B^2_{1/2}\times B^{n-2}_{3/4}} \frac{x_1^2+x_2^2}{2} e_\varepsilon(u,\nabla)\Delta\phi^2 \\
&\quad + C\int_{B^2_{1/2}\times B^{n-2}_{3/4}} \frac{x_1^2+x_2^2}{2} \sum_{3\leqslant j,k\leqslant n} (\nabla u^*\nabla u + \varepsilon^2\omega^*\omega)(e_j,e_k)\partial^2_{e_j,e_k}\phi^2 \\
&\quad + C\|D^2\phi\|_{L^\infty} e^{-K/\varepsilon}.
\end{aligned}$$

Now, by the soft height bound, we can assume that the vorticity set Z intersects $B^2_{1/2} \times B^{n-2}_{3/4}$ in a small cylinder $B^2_\sigma \times B^{n-2}_{3/4}$; the conclusion follows by exponential decay, up to replacing K with another constant $K(n, \sigma)$. ◻

Remark 16.2.2. In the statement of Proposition 16.2.1 we can replace the first term of the right-hand side as follows:

$$\int_{B^{n-2}_{3/4}} \phi^2(z)(\mathbf{E}_1)_z \, dz \leq C \int_{B^2_{1/2} \times B^{n-2}_{3/4}} [(x_1 - c_1)^2 + (x_2 - c_2)^2] e_\varepsilon(u, \nabla) \Delta\phi^2$$
$$+ C(\sigma^2 \mathbf{E}_1 + e^{-K/\varepsilon}) \|D^2\phi\|_\infty,$$

provided that $|c| \leq C\sigma$ for $C = C(n)$. The proof is essentially the same.

Chapter 17

Proof of decay of the tilt-excess

In this section we prove Theorem 10.3.2: roughly speaking, we prove that $\mathbf{E}_1$, the first part of the excess, decays up to scales where it becomes comparable with ε^2. We will deduce this from the excess decay property of harmonic functions, stated in the next elementary lemma.

Lemma 17.0.1. *Given a harmonic function $w : B_1^n(0) \to \mathbb{R}$, we have the decay estimate*

$$\sup_{x \in B_\rho^n(0)} |w(x) - w(0) - dw(0)[x]| \leq C(n)\rho^2 \|dw\|_{L^2}, \tag{17.1}$$

for $\rho \in (0, \frac{1}{2})$.

Proof. By a Taylor expansion, the left-hand side is bounded by the quantity $\frac{\rho^2}{2}\|D^2 w\|_{L^\infty(B^n_{1/2})}$, which is bounded by $C(n)\rho^2\|dw\|_{L^2}$ by the mean-value property of harmonic functions. □

17.1 Proof of the excess decay in the case of small $|dw(0)|$

First, we prove Theorem 10.3.2 when the harmonic approximation has $|dw(0)| \leqslant \delta$, for a small $\delta > 0$ to be chosen later. We dilate the ball B_1^n to $B^n_{\sqrt{2}}$ (and replace ε with $\varepsilon/\sqrt{2}$), in such a way that it includes $B_1^2 \times B_1^{n-2}$; we also assume that $S = \mathbb{R}^{n-2}$ in the statement.

Let c be the average of h on the ball $B^{n-2}_{3/4}$. The construction of h shows that

$$|c| \leqslant C\sigma + Ce^{-K/\varepsilon} \leqslant C\sigma$$

for ε small enough (depending on n, σ).

We apply the Caccioppoli-type estimates in Proposition 16.2.1, with $x_1 - c_1$ and $x_2 - c_2$ in place of x_1 and x_2, see Remark 16.2.2. Taking $\phi \in C^\infty_c(B^{n-2}_{2\rho})$ to be a cut-off function with $\phi = 1$ on B^{n-2}_ρ and $|D^2\phi| \leq C(n)\rho^{-2}$ we get

$$\begin{aligned}
&\int_{B^{n-2}_{3/4}} \phi^2(z)(\mathbf{E}_1)_z \, dz \\
&\leq C \int_{B^2_{1/2}\times B^{n-2}_{3/4}} e_\varepsilon(u,\nabla)[(x_1 - c_1)^2 + (x_2 - c_2)^2]\Delta\phi^2 \\
&\quad + C\rho^{-2}(\sigma^2 \mathbf{E}_1 + e^{-K/\varepsilon}).
\end{aligned}$$

The contribution of the bad set $B^{n-2}_{3/4} \backslash \mathcal{G}^\eta$ can be bounded using the soft height bound of Lemma 14.3.2 and energy estimate on the bad set (item (iii) in Proposition 15.2.1), obtaining

$$\begin{aligned}
&\int_{B^2_{1/2}\times(B^{n-2}_{3/4}\backslash \mathcal{G}^\eta)} e_\varepsilon(u,\nabla)[(x_1 - c_1)^2 + (x_2 - c_2)^2]\Delta\phi^2 \\
&\leqslant C\rho^{-2}(\sigma^2 + e^{-K/\varepsilon})\frac{\mathbf{E}_1}{\eta^2}.
\end{aligned}$$

On the good set $\mathcal{G}^\eta$, we apply Lemma B.0.2 to estimate the *second moment* of good slices as follows:

$$\begin{aligned}
&\left| \int_{B^2_{1/2}\times \mathcal{G}^\eta} e_\varepsilon(u,\nabla)[(x_1 - c_1)^2 + (x_2 - c_2)^2]\Delta\phi^2 - \int_{\mathcal{G}^\eta} \varepsilon^2 v_0 \, \Delta\phi^2 \right| \\
&\leqslant C\rho^{-2}\left[\varepsilon^2 |\log \mathbf{E}_2|^2 \mathbf{E}_2^{1/2} + \sigma^2 \mathbf{E}_1 + \int_{\mathcal{G}^\eta} |h - c|^2 + e^{-K\sigma/\varepsilon}\right]
\end{aligned} \tag{17.1}$$

(see also Remark B.0.3), where h is the Lipschitz approximation obtained in Proposition 15.2.1. Note that the term containing v_0 disappears once integrated on $B^{n-2}_{2\rho}$, as v_0 is a constant and $\Delta\phi^2$ has zero integral.

Combining the previous bounds, we arrive at

$$\begin{aligned}
&\left|\int_{B^{n-2}_{3/4}} \phi^2(z)(\mathbf{E}_1)_z\,dz\right| \\
&\leqslant C\rho^{-2}\int_{B^{n-2}_{2\rho}} |h-c|^2 \\
&\quad + C\rho^{-2}\left[(\sigma^2+\varepsilon^2)\frac{\mathbf{E}_1}{\eta^2} + \left(1+\frac{\mathbf{E}_1}{\eta^2}\right)e^{-K/\varepsilon} + \varepsilon^2|\log \mathbf{E}_2|^2\mathbf{E}_2^{1/2}\right].
\end{aligned}$$

Assuming $e^{-K/\varepsilon} \leqslant \mathbf{E}_1$, we now apply Proposition 16.1.2 and Lemma 17.0.1. Since $\|(h-c)-\sqrt{\mathbf{E}_1}w\|^2_{L^2} \leqslant \nu\mathbf{E}_1$, we have

$$\int_{B^{n-2}_{2\rho}} |h-c|^2 \leqslant 2\nu\mathbf{E}_1 + 2\mathbf{E}_1\int_{B^{n-2}_{2\rho}} |w|^2 \leqslant 2\nu\mathbf{E}_1 + C\mathbf{E}_1(\rho^{4+(n-2)} + \delta^2\rho^{2+(n-2)}).$$

Thus, for some $C = C(n)$ and $K = K(n,\sigma)$, we get

$$\begin{aligned}
&\rho^{2-n}\int_{B^{n-2}_{\rho}} \mathbf{E}_1(z) \\
&\leqslant C\mathbf{E}_1(\rho^{-n}\nu + \rho^2 + \delta^2) \\
&\quad + C\rho^{-n}\left[(\sigma^2+\varepsilon^2)\frac{\mathbf{E}_1}{\eta^2} + \left(1+\frac{\mathbf{E}_1}{\eta^2}\right)e^{-K/\varepsilon} + \varepsilon^2|\log \mathbf{E}_2|^2\mathbf{E}_2^{1/2}\right].
\end{aligned}$$

We now choose η, ρ and, *subsequently*, δ, σ, ν to be small enough. The claim follows (with the same plane $\overline{S} = \mathbb{R}^{n-2}$) once we assume that $\mathbf{E}_1$ is small enough.

17.2 Tilting the picture

In the general case, before using the Caccioppoli-type estimate, we need to tilt the picture slightly to ensure that $|dw|(0)$ is small enough. We assume that $\mathbb{R}^{n-2}$ minimizes $\mathbf{E}_1(u, \nabla, B_1^n, \cdot)$.

Consider a rotation $R \in SO(n)$ bringing $\mathbb{R}^{n-2}$ to the graph of the linear map $\sqrt{\mathbf{E}_1} dw(0)$. Since w is harmonic with the bound $\|dw\|_{L^2} \leq C$, we have $|\sqrt{\mathbf{E}_1} dw(0)| \leqslant C\sqrt{\mathbf{E}_1}$. Hence, we can find R such that

$$\|R - I\| \leq C\mathbf{E}_1^{1/2}, \quad \|(P_{\mathbb{R}^{n-2}} \circ R - I) \circ P_{\mathbb{R}^{n-2}}\| \leq C\mathbf{E}_1, \tag{17.1}$$

for a dimensional constant $C = C(n)$: indeed, calling S the graph of $\sqrt{\mathbf{E}_1} dw(0)$, using the spectral theorem we can find an orthonormal basis $\{v_3, \ldots, v_n\}$ of $\mathbb{R}^{n-2}$ such that the vectors $P_S(v_i)$ form an orthogonal basis of S, so that $\langle P_S(v_i), v_j\rangle = \langle P_S(v_i), P_S(v_j)\rangle = 0$ for $i \neq j$. Thus, $P_{\mathbb{R}^{n-2}} \circ P_S(v_i)$ is parallel to v_i and

$$\frac{P_S(v_i)}{|P_S(v_i)|} = \frac{(\sqrt{\mathbf{E}_1} dw(0)[v_i], v_i)}{\sqrt{1 + \mathbf{E}_1 |dw(0)[v_i]|^2}}$$

(under the identification $\mathbb{R}^{n-2} = \{0\} \times \mathbb{R}^{n-2}$).

We extend $\{v_3, \ldots, v_n\}$ to an orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$. The desired rotation is obtained by sending v_i to $\frac{P_S(v_i)}{|P_S(v_i)|}$ for $i \geqslant 3$ and v_1, v_2 to suitable unit vectors $v_1 + O(\sqrt{\mathbf{E}_1})$ and $v_2 + O(\sqrt{\mathbf{E}_1})$, obtained for instance via the Gram–Schmidt algorithm on the collection $\{\frac{P_S(v_3)}{|P_S(v_3)|}, \ldots, \frac{P_S(v_n)}{|P_S(v_n)|}, v_1, v_2\}$. For $i \geqslant 3$, since $|P_{S^\perp} v_i| \leqslant C\sqrt{\mathbf{E}_1}$ we have $|P_S v_i| \geqslant 1 - C\mathbf{E}_1$, and hence the previous formula gives

$$R(v_i) = R(0, v_i) = (\sqrt{\mathbf{E}_1} dw(0)[v_i], v_i) + O(\mathbf{E}_1) \quad \text{for } i \geqslant 3. \tag{17.2}$$

Then we define the rotated pair $(\tilde{u}, \tilde{\nabla})$ as follows:

$$\tilde{u} := R^* u, \quad \tilde{\nabla} := R^* \nabla. \tag{17.3}$$

First we prove that the excess changes proportionally after this rotation.

Lemma 17.2.1 (Tilted excess estimate)**.** *There exists a dimensional constant $C(n)$ such that, for a pair (u, ∇) as in Theorem 10.3.2 with small enough $\tau_0, \varepsilon_0 > 0$ and a rotation R as above, the tilted excess is bounded by the initial excess; more precisely,*

$$\boldsymbol{E}_1(\tilde{u}, \tilde{\nabla}, B_1^n, \mathbb{R}^{n-2}) \leq C\boldsymbol{E}_1, \quad \boldsymbol{E}_2(\tilde{u}, \tilde{\nabla}, B_1^n, \mathbb{R}^{n-2}) \leq C\boldsymbol{E}. \tag{17.4}$$

Proof. Take an orthonormal basis $e_1, e_2, \ldots, e_n$ for $\mathbb{R}^n$ such that $\{e_3, \ldots, e_n\}$ is an orthonormal basis for $\mathbb{R}^{n-2}$. Then, recalling the definition of the excess $\mathbf{E}_1$, we have

$$\begin{aligned}\mathbf{E}_1(\tilde{u}, \tilde{\nabla}, B_1^n, \mathbb{R}^{n-2}) &= \int_{B_1^n} \left[\sum_{k=3}^{n} |\nabla_{Re_k} u|^2 + \sum_{(j,k)\neq(1,2)} \varepsilon^2 \omega(Re_j, Re_k)^2 \right] \\ &\leq C\mathbf{E}_1 + C\|R - I\|^2 E_\varepsilon(u, \nabla) \\ &\leq C\mathbf{E}_1.\end{aligned}$$

The second line above follows from the elementary bounds

$$|\nabla_{Re_k} - \nabla_{e_k} u| \leq \|R - I\| |\nabla u|$$

and

$$|\omega(Re_j, Re_k) - \omega(e_j, e_k)| \leq 2\|R - I\| |\omega|.$$

We estimate $\mathbf{E}_2$ in a similar way:

$$\begin{aligned}
&\mathbf{E}_2(\tilde{u}, \tilde{\nabla}, B_1^n, \mathbb{R}^{n-2}) \\
&= \int_{B_1^n} \left[|\nabla_{Re_1} u + i\nabla_{Re_2} u|^2 + \left| \varepsilon\omega(Re_1, Re_2) - \frac{1-|u|^2}{2\varepsilon} \right|^2 \right] \\
&\leq C\mathbf{E}_2 + C\|R - I\|^2 E_\varepsilon(u, \nabla) \\
&\leq C(\mathbf{E}_1 + \mathbf{E}_2).
\end{aligned}$$

This is indeed the desired conclusion. □

Then we claim that the Lipschitz approximations h and $\tilde{h}$ are approximately a rotation of one another. To do this, we first notice that the Lipschitz approximation h_0 of the zero set in Proposition 15.3.1 (applied with $\delta = \sigma$) behaves well under rotations: take $\tilde{h}_0$ to be the function whose graph is obtained by rotating of the graph of h_0 by R^{-1} (cf. [80, Section 8.2]). For z in the domain of $\tilde{h}_0$, there exists $z' \in B_{3/4}^{n-2}$ such that

$$(\tilde{h}_0(z), z) = R^{-1}(h_0(z'), z').$$

Since $\|(P_{\mathbb{R}^{n-2}} \circ R - I) \circ P_{\mathbb{R}^{n-2}}\| \leq C\mathbf{E}_1$ and $|h_0| \leqslant \sigma$, we have $|z' - z| \leqslant C\mathbf{E}_1 + C\sqrt{\mathbf{E}_1}\sigma$. Moreover, we have $\mathrm{Lip}(h_0) \leqslant \sigma$, giving $|h_0(z') - h_0(z)| \leqslant C\sqrt{\mathbf{E}_1}\sigma$. Thus, assuming $\sqrt{\mathbf{E}_1} \leqslant \sigma$,

$$(\tilde{h}_0(z), z) = R^{-1}(h_0(z), z) + O(\sqrt{\mathbf{E}_1}\sigma);$$

recalling (17.2), we see that $R(0, z) = (\sqrt{\mathbf{E}_1} dw(0)[z], z) + O(\mathbf{E}_1|z|)$, so that

$$\tilde{h}_0(z) = h_0(z) - \sqrt{\mathbf{E}_1} dw(0)[z] + O(\sqrt{\mathbf{E}_1}\sigma), \tag{17.5}$$

with an implicit constant $C(n)$. Note that $\tilde{h}_0$ can be taken as a Lipschitz approximation of the zero set of the tilted pair: in order to have the conclusion of Proposition 15.3.1, the only property that we care about is that its graph covers the zeros of $\tilde{u}$, except some exceptional ones projecting on a set of measure at most $C(n)\frac{\mathbf{E}_1}{\eta^2}$, and this holds for the rotated graph.

17.3 Proof of the excess decay in the general case

Now we can use (17.5) and the L^2 bound from Proposition 15.3.1 to conclude the proof of the tilt-excess decay theorem in the general case.

Proof of Theorem 10.3.2. Recall that $\mathbb{R}^{n-2}$ minimizes $\mathbf{E}_1(u, \nabla, B_1^n, \cdot)$. Let $\tilde{h}$ be the Lipschitz approximation of the barycenter (built in Proposition 15.2.1) for the tilted pair $(\tilde{u}, \tilde{\nabla})$. We have

$$
\begin{aligned}
&|\tilde{h}(z) - (h(z) - \sqrt{\mathbf{E}_1} dw(0)[z])| \\
&\leq |\tilde{h} - \tilde{h}_0| + |h - h_0| + |\tilde{h}_0 - (h_0 - \sqrt{\mathbf{E}_1} dw(0)[z])|.
\end{aligned}
$$

We combine the main estimate of Proposition 15.3.1 and (17.4)–(17.5) to see that

$$
\begin{aligned}
&\int_{B_{1/2}^{n-2}} |\tilde{h}(z) - (h(z) - {\mathbf{E}_1}^{1/2} dw(0)[z])|^2 \\
&\leq C\left(\frac{\sigma^2}{\eta^2} + \sigma^2\right)\mathbf{E}_1 + C\varepsilon^2|\log \mathbf{E}|^2\mathbf{E} + Ce^{-K/\varepsilon}.
\end{aligned}
\tag{17.1}
$$

We assume in the sequel that

$$
\varepsilon^2|\log \mathbf{E}|^2\mathbf{E}, e^{-K/\varepsilon} \leqslant \sigma^2\mathbf{E}_1,
$$

so that

$$
\int_{B_{1/2}^{n-2}} |\tilde{h}(z) - (h(z) - \mathbf{E}_1^{1/2} dw(0)[z])|^2 \leq C\left(\frac{\sigma^2}{\eta^2} + \sigma^2\right)\mathbf{E}_1.
$$

Now, taking the harmonic approximation for the tilted pair to be $\tilde{w}$, we can see that

$$\begin{aligned}
&\int_{B^{n-2}_{1/2}} |\tilde{\mathbf{E}}_1^{1/2}\tilde{w}(z) - \mathbf{E}_1^{1/2}(w(z) - dw(0)[z])|^2\, dz \\
&\leq C \int_{B^{n-2}_{1/2}} [|h - \mathbf{E}_1^{1/2} w|^2 + |\tilde{h} - \tilde{\mathbf{E}}_1^{1/2}\tilde{w}|^2 + |\tilde{h}(z) - (h(z) - \mathbf{E}_1^{1/2} dw(0)[z])|^2] \\
&\leqslant C \left(\nu + \frac{\sigma^2}{\eta^2} + \sigma^2 \right) \mathbf{E}_1
\end{aligned}$$

(the last line follows from $\tilde{\mathbf{E}}_1 \leq C\mathbf{E}_1$, as we saw in (17.4)). Since

$$\tilde{\mathbf{E}}_1^{1/2}\tilde{w}(z) - \mathbf{E}_1^{1/2}(w(z) - dw(0)[z])$$

is harmonic, its differential at the origin

$$|\tilde{\mathbf{E}}_1^{1/2} d\tilde{w}(0)|^2 \leqslant C \left(\nu + \frac{\sigma^2}{\eta^2} + \sigma^2 \right) \mathbf{E}_1.$$

Since $\tilde{\mathbf{E}}_1 \geqslant \mathbf{E}_1$, this tells us that $|d\tilde{w}(0)|$ can be made arbitrarily small, reducing to the previous situation. □

Remark 17.3.1. In all the results obtained so far we were assuming that the center of the ball (or cylinder) belongs to the zero set, but actually they also hold if it belongs to the vorticity set $Z = \{|u| \leqslant \frac{3}{4}\}$, since this is enough to guarantee that it belongs to the support of the energy concentration measure in compactness arguments.

Chapter 18

Iteration arguments and Morrey-type bounds

18.1 Proof of Theorem 10.3.7: the case of critical pairs for $2 \leq n \leq 4$

We prove the following theorem, which is the first part of Theorem 10.3.7.

Theorem 18.1.1. *For $2 \leq n \leq 4$, there exists $\tau_0(n) > 0$ such that the following holds. If (u, ∇) is an entire critical point for the energy E_1, given by (12.1) for $\varepsilon = 1$, with $u(0) = 0$ and the energy bound*

$$\lim_{R \to \infty} \frac{1}{|B_R^{n-2}|} \int_{B_R^n} \left[|\nabla u|^2 + |F_\nabla|^2 + \frac{1}{4}(1 - |u|^2)^2 \right] \leq 2\pi + \tau_0, \tag{18.1}$$

then (u, ∇) is two-dimensional. More precisely, we have $(u, \nabla) = P^*(u_0, \nabla_0)$ up to a change of gauge, where P is the orthogonal projection onto a two-dimensional subspace and (u_0, ∇_0) is the standard degree-one solution of Taubes [74] (or its conjugate), centered at the origin.

Proof. We can assume $n \in \{3,4\}$. First, we claim that it is enough to show that

$$\lim_{R\to\infty} R^2 \min_S \mathbf{E}_1(u,\nabla,B_R^n,S) = 0.$$

Indeed, once this is done, we have

$$R^{4-n}\int_{B_R^n}\left[\sum_{a=3}^n |\nabla_{e_a^R}u|^2 + \sum_{(a,b)\neq(1,2)} \omega(e_a^R,e_b^R)^2\right] \to 0$$

as $R\to\infty$, for a suitable choice of planes $S(R)$, where $\{e_1^R,\dots,e_n^R\}$ is an orthonormal basis such that $S(R)$ is spanned by $\{e_3^R,\dots,e_n^R\}$. Extracting a limit $S(R)\to S$ along a subsequence and assuming without loss of generality that $S=\mathbb{R}^{n-2}$, the fact that $n\leqslant 4$ and Fatou's lemma give

$$\int_{\mathbb{R}^n}\left[\sum_{a=3}^n |\nabla_{e_a}u|^2 + \sum_{(a,b)\neq(1,2)} \omega(e_a,e_b)^2\right] = 0.$$

As in the proof of Lemma 14.3.4, this implies that (u,∇) depends only on the first two coordinates up to a change of gauge, and the conclusion follows from the classification of planar solutions by Taubes [74].

We now turn to the previous claim. By Proposition 14.3.1, we have

$$\frac{1}{|B_R^{n-2}|}\int_{B_R^n} e_\varepsilon(u,\nabla) \to 2\pi$$

as $R\to\infty$, as well as

$$\mathbf{E}(u,\nabla,B_R^n,S(R)) \to 0$$

for suitable oriented planes $S(R)$, up to conjugating the pair. Arguing as in the proof of Proposition 14.3.1, we see that $S(R)$, viewed as an unoriented plane, has vanishing distance from any unoriented plane S minimizing $\mathbf{E}_1(u,\nabla,B_R^n,S)$; hence, we can assume that $S(R)$ minimizes $\mathbf{E}_1$ on

B^n_R.

The proof now becomes an elementary iteration argument. In Theorem 10.3.2 we first fix $\rho \in (0,1)$ such that $C\rho^2 \leqslant \rho$ and then τ and ε_0 accordingly. Let $C' > \frac{1}{\varepsilon_0}$. Without loss of generality we can also assume that

$$\mathbf{E}_1(u,\nabla,B^n_R,S(R)) > 0, \quad \mathbf{E}(u,\nabla,B^n_R,S(R)) \in (0,1)$$

are small enough to allow applying Theorem 10.3.2 on B^n_R (by rescaling our pair), for all $R \geqslant C'$. For every $k \in \mathbb{N}$ let us define the minimum excess on each ball $B_{C'\rho^{-k}}$:

$$\mathbf{E}_1(k) := \mathbf{E}_1(u,\nabla,B_{C'\rho^{-k}},S(C'\rho^{-k})).$$

Then Theorem 10.3.2 gives

$$\begin{aligned} &\text{either } \mathbf{E}_1(k) \leq \rho\overline{\mathbf{E}}_1(k+1) \\ &\quad \text{or } \mathbf{E}_1(k) \leq \max\{\rho^{2k}|\log \mathbf{E}(k+1)|^2\sqrt{\mathbf{E}(k+1)}, e^{-K\rho^{-2k}}\}, \end{aligned} \tag{18.2}$$

where $\mathbf{E}(k) := \mathbf{E}(u,\nabla,B_{C'\rho^{-k}},S(C'\rho^{-k}))$. By Proposition 14.3.1, we have

$$\lim_{k\to\infty} \mathbf{E}_1(k) = 0, \quad \lim_{k\to\infty} \mathbf{E}(k) = 0. \tag{18.3}$$

In order to iterate (18.2), we define

$$f(k) := \log \mathbf{E}_1(k) + 2k \log \rho^{-1}$$

and

$$g(k) := \max\left\{2\log|\log \mathbf{E}(k+1)| + \frac{1}{2}\log \mathbf{E}(k+1), -K\rho^{-2k} + 2k\log\rho^{-1}\right\}.$$

Then (18.2) can be rewritten in terms of the functions $f, g : \mathbb{N} \to \mathbb{R}$ as

$$f(k) \leq f(k+1) - \lambda \quad \text{or} \quad f(k) \leq g(k), \tag{18.4}$$

where $\lambda := 3 \log \rho^{-1}$. Condition (18.3) also means that

$$\lim_{k\to\infty} f(k) - 2k \log \rho^{-1} = -\infty, \quad \lim_{k\to\infty} g(k) = -\infty. \tag{18.5}$$

We claim that if f, g satisfy (18.4) and (18.5) then

$$f(k) \leq \sup_{m \geq k} [g(m) - \lambda(m-k)].$$

We prove this by contradiction: assume that there is some index k_0 such that

$$f(k_0) + \lambda(m - k_0) > g(m) \quad \text{for all } m \geq k_0. \tag{18.6}$$

In particular we have $f(k_0) > g(k_0)$, so that (18.4) and (18.6) give

$$f(k_0 + 1) \geq f(k_0) + \lambda > g(k_0 + 1).$$

By induction, we see that for all $m \geq k_0$

$$f(m) \geq f(k_0) + \lambda(m - k_0).$$

Taking the limit $m \to \infty$ and noting that $\lambda > 2\log\rho^{-1}$, we obtain

$$\begin{aligned} f(k_0) &\leq \lim_{m\to\infty}[f(m) - \lambda(m-k_0)] \\ &\leq \lim_{m\to\infty}[f(m) - 2m\log\rho^{-1}] + 2k_0\log\rho^{-1} \\ &= -\infty, \end{aligned}$$

where we used (18.5) in the last equality. This is a contradiction, proving our claim.

As a consequence, we have

$$f(k) \leq \sup_{m\geq k}[g(m) - \lambda(m-k)] \leq \sup_{m\geq k} g(m).$$

Since $\lim_{k\to\infty} g(k) = -\infty$ by (18.5), we deduce that

$$\lim_{k\to\infty} f(k) = -\infty.$$

In other words, we have $\rho^{-2k}\mathbf{E}_1(k) \to 0$, as desired. □

18.2 Proof of Corollary 10.3.4 and Theorem 10.3.5

Given any $n \geqslant 3$ and (u, ∇) as in Theorem 10.3.5, for any $\tau_0' > 0$ a standard compactness argument shows that

$$\frac{1}{|B_r^{n-2}|}\int_{B_r(x)} e_\varepsilon(u,\nabla) \leqslant 2\pi + \tau_0'$$

for all $x \in Z \cap B^n_{3/4}$ and $r = \frac{1}{8}$, provided that τ_0 and ε_0 are taken small enough, and hence also for $r \leqslant \frac{1}{7}$ by energy monotonicity.

This, together with Proposition 14.3.1, implies that, for some oriented planes $S(x,r)$,

$$\mathbf{E}(u,\nabla,B_r(x),S(x,r)) \leqslant \delta$$

for some $\delta > 0$ to be chosen momentarily and $C(n,\delta)\varepsilon \leqslant r \leqslant \frac{1}{8}$. As in the previous proof, we can assume that $S(x,r)$ minimizes $\mathbf{E}_1$ on the ball $B_r(x)$.

Given $\alpha \in [0,1)$, we first fix ρ such that $C\rho^2 \leqslant \rho^{2\alpha}$ where C is the constant appearing in Theorem 10.3.2. We now choose δ, τ_0 small such that Theorem 10.3.2 applies on each ball $B_r(x)$ with $x \in Z \cap B^n_{3/4}$, compare with Remark 17.3.1. We then consider

$$\max\{M\varepsilon, \varepsilon^{1/(1+\alpha)}\} \leqslant r \leqslant \frac{1}{8} \tag{18.1}$$

where M chosen large enough to ensure that

$$e^{-Kr/\varepsilon} \leqslant \frac{\varepsilon^2}{r^2}$$

if $\varepsilon/r \leqslant 1/M$. Applying the scaled version of Theorem 10.3.2 (with ε replaced by ε/r), and noticing that $\sup_{0<s\leqslant\delta} |\log\delta|^2\delta^{1/2} \leqslant 1$, we finally obtain that either

$$\mathbf{E}_1(u,\nabla,B_{\rho r}(x),S(x,\rho r)) \leqslant \rho^{2\alpha}\mathbf{E}_1(u,\nabla,B_r(x),S(x,r))$$

or

$$\mathbf{E}_1(x,r) := \mathbf{E}_1(u,\nabla,B_r(x),S(x,r)) \leqslant \frac{\varepsilon^2}{r^2} \leqslant r^{2\alpha}.$$

This immediately implies

$$\mathbf{E}_1(x,r) \leqslant C(n,\alpha)r^{2\alpha} \quad \text{for all } x \in Z \cap B^n_{3/4} \text{ and any radii satisfying (18.1).}$$

Moreover, if $S(x,r)$ is different from $S(x,r')$, for some $r' \in [r,2r]$, then we can find an orthonormal basis $\{e_1,\dots,e_n\}$ such that $\{e_3,\dots,e_n\}$ spans $S(x,r)$ and e_2 belongs to the span of the two planes, namely $e_2 = v + w$ with $v \in S(x,r)$ and $w \in S(x,r')$, with the bound

$$|v| + |w| \leqslant C(n)\|P_{S(x,r)} - P_{S(x,r')}\|^{-1},$$

as the next simple lemma shows.

Lemma 18.2.1. *Given two different planes $S, S' \in \mathrm{Gr}(n,k)$, there exists a unit vector $e \in (S+S')\cap S^\perp$ such that $e = v + w$, with $v \in S$, $w \in S'$, and $|v|, |w| \leqslant C(n)\|P_S - P_{S'}\|^{-1}$.*

Proof. We can assume that $S + S' = \mathbb{R}^n$ and $S \cap S' = \{0\}$ (otherwise we work on $(S \cap S')^\perp$), so that $n = 2k$. We can also assume without loss of generality that $\|P_S - P_{S'}\|_{op} < c(n)$ for a constant $c(n) > 0$ to be determined momentarily, since otherwise the statement follows from an immediate compactness argument (using the fact that, if $S_j \to S_\infty$ and $S'_j \to S'_\infty$, then each unit vector in $S_\infty + S'_\infty$ has vanishing distance from $S_j + S'_j$, even when the former sum has smaller dimension).

It is elementary to check that the statement holds when $k = 1$: in this case, calling $\theta \in (0, \frac{\pi}{2}]$ the angle between the lines S and S', we have $\|P_S - P_{S'}\| = \sqrt{2}\sin\theta$, and we can find vectors as in the statement with $|v|, |w| \leqslant \frac{1}{\sin\theta}$.

Let $\tilde{e}$ be an eigenvector of $P_S - P_{S'}$, corresponding to an eigenvalue λ with $0 < |\lambda| = \|P_S - P_{S'}\|_{op} < c(n)$. Then

$$P_S\tilde{e} - P_{S'}\tilde{e} = \lambda\tilde{e},$$

so that in particular $P_S\tilde{e}, P_{S'}\tilde{e} \neq 0$ and

$$P_S P_{S'}\tilde{e} = (1-\lambda)P_S\tilde{e}.$$

Similarly we have

$$P_{S'}P_S\tilde{e} = (1+\lambda)P_{S'}\tilde{e}.$$

From the equation

$$\langle P_{S'}P_S\tilde{e}, P_{S'}\tilde{e}\rangle = \langle P_S\tilde{e}, P_{S'}\tilde{e}\rangle = \langle P_S\tilde{e}, P_SP_{S'}\tilde{e}\rangle$$

we deduce that

$$|P_{S'}\tilde{e}|^2 = \frac{1-\lambda}{1+\lambda}|P_S\tilde{e}|^2.$$

In particular, calling $\theta \in (0, \frac{\pi}{2}]$ the angle between the vectors $P_S\tilde{e}$ and $P_{S'}\tilde{e}$, these identities easily give

$$\sin^2\theta = 1 - \frac{\langle P_S\tilde{e}, P_{S'}\tilde{e}\rangle^2}{|P_S\tilde{e}|^2|P_{S'}\tilde{e}|^2} = \lambda^2.$$

We now take

$$Z := \mathrm{span}\{P_S\tilde{e}, P_{S'}\tilde{e}\},$$

which is a two-dimensional plane. By the case $k = 1$, we can find

$$e \in Z, \quad v \in \mathrm{span}\{P_S\tilde{e}\}, \quad w \in \mathrm{span}\{P_{S'}\tilde{e}\}$$

such that $e \perp P_S\tilde{e}$ is a unit vector and $|v|, |w| \leqslant \lambda^{-1}$. In order to conclude, it suffices to check that $e \perp S$. Writing

$$e = \alpha P_S\tilde{e} + \beta P_{S'}\tilde{e},$$

we have

$$P_Se = \alpha P_S\tilde{e} + \beta P_SP_{S'}\tilde{e} = [\alpha + \beta(1-\lambda)]P_S\tilde{e}.$$

Since $e \perp P_S\tilde{e}$, we have

$$0 = \langle \alpha P_S\tilde{e} + \beta P_{S'}\tilde{e}, P_S\tilde{e}\rangle = \alpha|P_S\tilde{e}|^2 + \beta\langle P_SP_{S'}\tilde{e}, P_S\tilde{e}\rangle = [\alpha + \beta(1-\lambda)]|P_S\tilde{e}|^2.$$

Since $P_S\tilde{e} \neq 0$, we have $\alpha + \beta(1-\lambda) = 0$, proving the claim. $\square$

Since $\mathbf{E}(x,r) \leqslant \delta$, we have

$$\begin{aligned} & r^{2-n}\int_{B_r(x)} e_\varepsilon(u,\nabla) \\ & \leqslant C(n)\delta + C(n)r^{2-n}\int_{B_r(x)}\left[\sum_{k=2}^n |\nabla_{e_k}u|^2 + \sum_{(j,k)}\varepsilon^2\omega(e_j,e_k)^2\right]. \end{aligned}$$

Since the left-hand side is close to 2π, and in particular larger than π (for $r \geqslant C\varepsilon$), using the previous fact from linear algebra for the term $\nabla_{e_2}u$ we obtain

$$1 \leqslant C(n)[\mathbf{E}_1(x,r) + \mathbf{E}_1(x,r')](\|P_{S(x,r)} - P_{S(x,r')}\|^{-2} + 1),$$

and thus, since $\|P_{S(x,r)} - P_{S(x,r')}\| \leqslant C(n)$,

$$\|P_{S(x,r)} - P_{S(x,r')}\| \leqslant C(n)\sqrt{\mathbf{E}_1(x,r) + \mathbf{E}_1(x,r')} \leqslant C(n,\alpha)r^\alpha. \tag{18.2}$$

As a consequence, summing over dyadic scales, we have

$$\|P_{S(x,r)} - P_{S(x,s)}\| \leqslant C(n,\alpha)\max\{r,s\}^\alpha$$

for $\max\{C(n,\alpha)\varepsilon, \varepsilon^{1/(1+\alpha)}\} \leqslant r,s \leqslant \frac{1}{8}$.

A similar argument works varying the center: for two different points $x, x' \in Z \cap B^n_{3/4}$, looking at the balls $B_r(x) \subset B_{2r}(x')$ with $r := |x-x'|$, we also have

$$\|P_{S(x,r)} - P_{S(x',r)}\| \leqslant C(n,\alpha)r^\alpha, \tag{18.3}$$

provided that $r = |x-x'| \in [\max\{C(n,\alpha)\varepsilon, \varepsilon^{1/(1+\alpha)}\}, \frac{1}{16}]$.

Actually, the previous proof gives some extra information, which will be crucial in the sequel. We record it in the next proposition.

Proposition 18.2.2. *Up to a rotation, we have $\|P_{S(x,r)} - P_{\mathbb{R}^{n-2}}\| \leqslant \gamma$ for any $\gamma > 0$ fixed in advance (up to decreasing ε_0, τ_0), for all $x \in Z \cap B^n_{3/4}$ and $r \in [C(n,\gamma)\varepsilon, \frac{1}{8}]$.*

Proof. Let C be the constant in the excess decay statement, fix ρ such that $C\rho^2 \leqslant \rho$ and fix τ_0 and ε_0 accordingly. Letting $r_k := \rho^k$, the first inequality of (18.2) gives that

$$\|P_{k+1} - P_k\| \leqslant C\sqrt{\mathbf{E}_1(x, r_k)}.$$

Iterating we get that

$$\begin{aligned}
\|P_\ell - P_k\| &\leqslant C(n)\sum_{j=k}^{\ell-1}\sqrt{\mathbf{E}_1(x, r_j)} \\
&\leqslant C(n)\sqrt{\mathbf{E}_1(x, r_k)}(1 + \rho^{1/2} + \rho + \dots) \\
&\leqslant C\sqrt{\mathbf{E}_1(x, r_k)}
\end{aligned}$$

as long as $\mathbf{E}_1(x, r_j) > \frac{\varepsilon^2}{r_j^2}$ for $j = 0, \dots, \ell - 1$ and $r_\ell \geqslant M\varepsilon =: \bar{r}$ where M is a large constant that we will fix at the end. Hence, if we call $r_{k_1} > \dots > r_{k_N} \geqslant \bar{r}$ the possible radii where $\mathbf{E}_1(x, r_{k_i}) \leqslant \frac{\varepsilon^2}{r_{k_i}^2}$, we deduce that

$$\begin{aligned}
\|P_\ell - P_0\| &\leqslant C\max\{\sqrt{\mathbf{E}_1(x, r_0)}, \sqrt{\mathbf{E}_1(x, r_{k_1})}, \dots, \sqrt{\mathbf{E}_1(x, r_{k_N})}\} \\
&\leqslant C\left[\sqrt{\mathbf{E}_1(x, r_0)} + \frac{\varepsilon}{\bar{r}}\right] \\
&\leqslant C\left[\sqrt{\mathbf{E}_1(x, r_0)} + \frac{1}{M}\right].
\end{aligned}$$

Also, P_0 can be assumed arbitrarily close to $\mathbb{R}^{n-2}$ by a simple compactness argument (similar to the proof of Proposition 14.3.1). Since $\sqrt{\mathbf{E}_1(x, r_0)}$ and $1/M$ can be taken arbitrarily small, the claim follows. ◻

The same proof gives the following.

Proposition 18.2.3. *For any $x \in Z \cap B^n_{3/4}$ and $r \in [C(n)\varepsilon, \frac{1}{8}]$, we have*

$$\boldsymbol{E}_1(u, \nabla, B_r(x), \mathbb{R}^{n-2}) \leqslant C(n)\boldsymbol{E}_1(u, \nabla, B_1(0), \mathbb{R}^{n-2}) + C(n)\frac{\varepsilon^2}{r^2}.$$

We now prove Corollary 10.3.4.

Proof of Corollary 10.3.4. We have already seen in Proposition 14.3.1 that the energy on B_R is asymptotic to $2\pi R^{n-2}$. We can then apply Proposition 18.2.2: for any $\gamma > 0$ we have

$$\|P_{S(0,R)} - P_{S(0,R')}\| \leqslant \gamma$$

for $R < R'$ large enough (we use Proposition 18.2.2 after scaling the picture down by a factor $(R')^{-1}$). We deduce that the limit

$$\lim_{R\to\infty} S(0, R)$$

exists. ◻

Proposition 18.2.4. *Up to a rotation, the vorticity set $\tilde{Z} := Z \cap [B^2_{1/2} \times B^{n-2}_{1/2}]$ is included in a $C(n,\gamma)\varepsilon$-neighborhood of the graph of a C^1 map*

$$f : B^{n-2}_{1/2} \to B^2_\gamma$$

with $\mathrm{Lip}(f) \leqslant \gamma$, if we assume that τ_0 and ε_0 are small enough (depending on n, γ).

Proof. Indeed, as seen in the proof of Proposition 14.3.1, for ε small enough we have $u(\cdot, z) \neq 0$ on $\partial B^2_{1/2}$, for all $z \in B^{n-2}_{1/2}$, and the degree of $(u/|u|)(\cdot, z)$ is ± 1 on this circle. Hence, each slice $B^2_{1/2} \times \{z\}$ intersects the zero set.

Moreover, using Lemma 14.3.2 on $B_1(0)$, we see that $\tilde{Z} \subseteq B^2_\gamma \times B^{n-2}_{1/2}$. Also, Lemma 14.3.2 implies that for all $x \in Z \cap B^n_{3/4}$ and $r \in [C(n,\gamma)\varepsilon, \frac{1}{8}]$ we have

$$Z \cap B_r(x) \subseteq B_{\gamma r}(x + S(x,r)), \tag{18.4}$$

where $B_{\gamma r}(x + S(x,r))$ is the γr-neighborhood of the affine plane $x + S(x,r)$. We now take a collection of points $\{z_k\} \subset B^{n-2}_{1/2}$ with pairwise distance at least $C(n,\gamma)\varepsilon$ and $B^{n-2}_{1/2} \subseteq \bigcup_k B^{n-2}_{5C(n,\gamma)\varepsilon}(z_k)$. For each k, we fix a point $x_k = (y_k, z_k) \in \tilde{Z}$. We then see that

$$|y_k - y_j| \leqslant C\gamma |x_k - x_j|,$$

thanks to the previous observation applied with $r := 2|x_k - x_j|$ and the fact that $S(x,r)$ is γ-close to $\mathbb{R}^{n-2}$ (for $|x_k - x_j| > \frac{1}{16}$, this follows just from Lemma 14.3.2). Hence, the assignment $z_k \mapsto y_k$ defines a $C(n)\gamma$-Lipschitz function, which we can extend to a $C(n)\gamma$-Lipschitz function $f : B^{n-2}_{1/2} \longrightarrow B^2_\gamma$. It is easy to check that (a regularization of) f satisfies the desired conclusion, completing the proof. □

We are now in position to prove Theorem 10.3.5.

Proof of Theorem 10.3.5. Let $\eta > 0$ small such that Proposition 15.2.1 applies. We first remark that the previous points x_k can be taken such that $u(x_k) = 0$ and $z_k \in \mathcal{G}^\eta$. Indeed, by Proposition 18.2.3, we have

$$\mathbf{E}_1(u, \nabla, B_r(x), \mathbb{R}^{n-2}) \leqslant c(n)\eta^2$$

for all points $x \in Z \cap B^n_{3/4}$ and radii $r \in [C(n)\varepsilon, \frac{1}{8}]$ (by taking ε_0, τ_0 suitably small). We can apply this with $r := M\varepsilon$; by Proposition 18.2.4 and exponential decay of energy away from Z, we have

$$r^{2-n} \int_{B^{n-2}_{r/2}(z)} (\mathbf{E}_1)_z \leqslant c(n)\eta^2 + e^{-K/M} \leqslant 2c(n)\eta^2$$

once we take $M = C(n)$ large enough, for any $z \in B^{n-2}_{1/2}$. Once we take $c(n)$ small enough, by the weak L^1 bound we can then find

$$z' \in B^{n-2}_{r/2}(z) \cap \mathcal{G}^\eta$$

(where we use slices of radius $\frac{1}{2}$ in the definition of $\mathcal{G}^\eta$), showing the claim.

As a consequence of Lemma B.0.1, we deduce that

$$|h(z_k) - y_k| = |h(z_k) - h_0(z_k)| \leqslant \varepsilon.$$

We immediately deduce that $\tilde{Z}$ is included in a $C(n)\varepsilon$-neighborhood of the graph of h, which is the only consequence of the claim which we will use in the sequel.

Now let $\overline{\rho} := \max\{M\varepsilon, \varepsilon^{1/(1+\alpha)}\}$ (with M as in (18.1)) and consider *another* finite collection of points $\{z_k\} \subset B^{n-2}_{1/2}$ such that the balls $B^{n-2}_{\overline{\rho}}(z_k)$ are disjoint and the dilated balls $B^{n-2}_{4\overline{\rho}}(z_k)$ cover $B^{n-2}_{1/2}$. Let $x_k = (y_k, z_k)$ be a point in $\tilde{Z}$ for each k.

On $B^n_{10\overline{\rho}}(x_k)$ we consider the Lipschitz approximation h_k built with respect to the rotated picture, obtained as a graph over $S_k := x_k + S(x_k, 10\overline{\rho})$. When viewed as a graph over $\mathbb{R}^{n-2}$, it becomes a function $\tilde{h}_k$ defined on the slightly smaller ball $B^{n-2}_{5\overline{\rho}}(z_k)$.

By a scaled version of Proposition 15.2.1, we have

$$\int_{B^n_{30\overline{\rho}/4}(x_k)\cap S_k} |dh_k|^2 \leqslant C\overline{\rho}^{n-2}\mathbf{E}_1(u, \nabla, B^n_{10\overline{\rho}}(x_k), S(x_k, 10\overline{\rho})) \leqslant C\overline{\rho}^{n-2+2\alpha}.$$

In particular, by Poincaré,

$$\int_{B^n_{30\overline{\rho}/4}(x_k)\cap S_k} |h_k - (h_k)|^2 \leqslant C\overline{\rho}^{n+2\alpha}.$$

Now, as in (17.5), we observe that

$$|\tilde{h}_k(z) - h_k(z) - A_k(z)| \leqslant C\overline{\rho}\sqrt{\mathbf{E}_1(u, \nabla, B^n_{10\overline{\rho}}(x_k), S(x_k, 10\overline{\rho}))} \leqslant C\overline{\rho}^{1+\alpha}$$

for a suitable affine function A_k (where, with abuse of notation, $h_k(z)$ means h_k composed with the isometry $\mathbb{R}^{n-2} \to S_k$). Combining these two bounds, we get

$$\int_{B^{n-2}_{5\overline{\rho}}(z_k)} |\tilde{h}_k - A'_k|^2 \leqslant C\overline{\rho}^{n+2\alpha}$$

for another affine function A'_k.

We now take a partition of unity φ_k subordinated to the cover $\{B^{n-2}_{4\overline{\rho}}(z_k)\}$ and we let

$$f := \sum_k \varphi_k \tilde{h}_k.$$

Since the zero set is within a $C\varepsilon$-neighborhood of the graph of $\tilde{h}_k$ (on the set $B^2_{1/2} \times B^{n-2}_{5\overline{\rho}}(z_k)$), we deduce that

$$|\tilde{h}_k - \tilde{h}_{k'}| \leqslant C\varepsilon$$

on $B^{n-2}_{5\overline{\rho}}(z_k) \cap B^{n-2}_{5\overline{\rho}}(z_{k'})$. Thus, we also have

$$|A_k - A_{k'}| \leqslant C\varepsilon + C\overline{\rho}^{1+\alpha}$$

whenever $B^{n-2}_{4\overline{\rho}}(z_k) \cap B^{n-2}_{4\overline{\rho}}(z_{k'}) \neq \emptyset$. Since $\varepsilon \leqslant \overline{\rho}^{1+\alpha}$, this allows us to conclude that

$$\int_{B^{n-2}_{2\overline{\rho}}(z)} |f - A_z|^2 \leqslant C\overline{\rho}^{n+2\alpha}$$

for any $z \in B^{n-2}_{1/2}$, for a suitable affine function A_z depending on z.

Thus, taking a standard mollifier $\chi_{\overline{\rho}}$, setting

$$g := \chi_{\overline{\rho}} * f$$

and using the previous bound, we deduce that

$$|dg - dA_z| \leqslant C\overline{\rho}^{\alpha} \quad \text{on } B^{n-2}_{\overline{\rho}}(z),$$

and in fact

$$[dg]_{C^{0,\alpha}(B^{n-2}_{\overline{\rho}}(z))} \leqslant C.$$

Finally, recalling that dA_k is the slope of the plane $P_{S(x_k,10\overline{\rho})}$, we also have

$$|dA_k - dA_{k'}| \leqslant C|z_k - z_{k'}|^{\alpha}$$

by the Hölder continuity (18.3), while

$$|dA_z - dA_k| \leqslant C\overline{\rho}^{\alpha} \quad \text{for } z \in B^{n-2}_{4\overline{\rho}}(z_k).$$

From these bounds, we easily deduce that

$$|dg(z) - dg(z')| \leqslant C|z - z'|^{\alpha} \quad \text{for } |z - z'| \geqslant \overline{\rho},$$

completing the proof of the $C^{1,\alpha}$ regularity of g. Since $|g - f| \leqslant C\overline{\rho}$, it follows that the vorticity set is included in a $C\overline{\rho}$-neighborhood of the graph of g.

It is clear from the proof that we can actually make $[dg]_{C^{0,\alpha}}$ arbitrarily small, up to decreasing τ_0 and ε_0. □

Chapter 19

Constructing competitors for local minimizers: a good gauge

In this section we prepare the ground to construct competitors for minimizers and to show that the full excess decays as long as it is above ε^{β}, for any $\beta > 0$, giving a proof of Theorem 10.3.8. To investigate minimizers, we construct competitors with the same boundary conditions and compare the energies to show that the excess **E** is effectively approximated by the Dirichlet energy of the harmonic approximation.

To do this, we need to construct competitors modeled on graphs in the interior and then glue them to the boundary condition, while controlling the error terms. We pullback the ε-rescaled degree-one solution along the graph of the Lipschitz approximation, as obtained in Proposition 15.2.1. Then we gauge fix in balls of size $\varepsilon|\ln \varepsilon|$ and, using the estimates at that scale, we define a global gauge by a partition of unity. In this gauge we can interpolate between the intial pair and the new one with good estimates.

19.1 The pullback pair

Here we introduce the pullback pair (u_f, ∇_f) whose zero set is prescribed to be the graph of a Lipschitz function $f : B^{n-2} \to B^2_1$. We prove that the excess of these pairs are well approximated by the Dirichlet energy of f, as in the following proposition.

Proposition 19.1.1 (Constructing the pullback pair)**.** *There exist small constants $\eta_0(n), \varepsilon_0(n) > 0$ with the following property. Given any $\varepsilon \leq \varepsilon_0$ and a Lipschitz function $f : B^{n-2}_1 \to B^2_{1/2}$ with $Lip(f) = \eta \leq \eta_0$, there exists a pair (u_f, ∇_f) obeying the following estimate:*

$$\frac{1}{2\pi}\int_{B^2_1 \times B^{n-2}_1} e_\varepsilon(u_f, \nabla_f) = |B^{n-2}_1| + (1 + O(\eta^2))\int_{B^{n-2}_1} \frac{|df|^2}{2} + O(e^{-K/\varepsilon}),$$

with implicit constants $C(n)$. Moreover, we have that

$$u_f^{-1}(0) = \mathrm{graph}(f).$$

Proof. To construct (u_f, ∇_f) we pull back the planar degree-one solution of Taubes [74], via the map $Q_\varepsilon : B^2_1 \times B^{n-2}_1 \to \mathbb{R}^2$ given by

$$Q_\varepsilon(x) = \frac{(x_1, x_2) - f(x_3, \ldots, x_n)}{\varepsilon}.$$

Then we define (u_f, ∇_f) by

$$(u_f, \nabla_f) := Q^*_\varepsilon(u_0, \nabla_0), \tag{19.1}$$

where (u_0, ∇_0) is the degree-one solution in [74] with $u_0(0) = 0$ (unique up to change of gauge).

First, we note that, since

$$dQ_\varepsilon(x)[e_k] = -\partial_{e_k} f_1(x_3, \ldots, x_n)e_1 - \partial_{e_k} f_2(x_3, \ldots, x_n)e_2,$$

we have the following identities for $k = 3, \ldots, n$:

$$\begin{aligned} |(\nabla_f)_k u_f|^2(x) &= \varepsilon^{-2}|\partial_{e_k} f_1 (\nabla_0)_{e_1} u_0 + \partial_{e_k} f_2 (\nabla_0)_{e_2} u_0|^2 \left(Q_\varepsilon(x)\right) \\ &= \varepsilon^{-2}\frac{|\partial_k f|^2}{2}|\nabla_0 u_0|^2 \left(Q_\varepsilon(x)\right), \end{aligned} \tag{19.2}$$

where we omitted the argument of f and we used the fact that $(\nabla_0)_{e_2} u_0 = i(\nabla_0)_{e_1} u_0$ for solutions of the vortex equations (13.2). We also have

$$|(\nabla_f)_{e_1} u_f|^2(x) + |(\nabla_f)_{e_2} u_f|^2(x) = \varepsilon^{-2}|\nabla_0 u_0|^2 \left(Q_\varepsilon(x)\right). \tag{19.3}$$

We also compute for the curvature term $-i\omega_f := F_{\nabla_f} = F_{Q_\varepsilon^*(\nabla_0)} = Q_\varepsilon^*(F_{\nabla_0})$ that

$$\sum_{j=1,2} \varepsilon^2 \omega_f(e_j, e_k)^2(x) = \varepsilon^{-2}\omega_0(e_1, e_2)^2 \left(Q_\varepsilon(x)\right) |\partial_{e_j} f|^2 \quad \text{for } j = 1, 2,\ k \geqslant 3, \tag{19.4}$$

and moreover

$$\varepsilon^2 \omega_f(e_1, e_2)^2(x) = \varepsilon^{-2}\omega_0^2(e_1, e_2) \left(Q_\varepsilon(x)\right), \tag{19.5}$$

as well as

$$\varepsilon^2 \omega_f(e_k, e_\ell)^2(x) \leq \varepsilon^{-2}|df|^4 \omega_0(e_1, e_2)^2 \left(Q_\varepsilon(x)\right) \quad \text{for } 3 \leq k < \ell \leqslant n. \tag{19.6}$$

To compute $E_\varepsilon(u_f, \nabla_f)$, we use (19.2)–(19.6) to see that

$$
\begin{aligned}
&\int_{B_1^2 \times B_1^{n-2}} e_\varepsilon(u, \nabla) \\
&= \int_{B_1^{n-2}} \left[\int_{B_1^2} \varepsilon^{-2} e_1(u_0, \nabla_0)(Q_\varepsilon(x)) \left(1 + \frac{|df|^2}{2} + O(|df|^4) \right) + O(e^{-K/\varepsilon}) \right] \\
&= 2\pi \left[|B_1^{n-2}| + \int_{B_1^{n-2}} \frac{|df|^2}{2} + O(|df|^4) \right] + O(e^{-K/\varepsilon}).
\end{aligned}
$$

In the above display we used the exponential decay from [51, Chapter III, Theorem 8.1]:

$$
\int_{\mathbb{R}^2 \setminus B^2_{1/(2\varepsilon)}} e_1(u_0, \nabla_0) = O(e^{-K/\varepsilon}).
$$

Recalling that $|df| \leq \eta$ we get the desired estimate. □

19.2 Constructing the interpolation gauge

In this section we find a gauge transformation $(u, \nabla) \mapsto (e^{i\xi}u, \nabla - id\xi)$ for which the new pair is L^2-close to the pullback pair (u_h, ∇_h) constructed in Proposition 19.1.1, where $h : B_1^{n-2} \to B^2_{1/2}$ is the Lipschitz approximation built in Proposition 15.2.1. Since this is the most technical part of the thesis, we provide an overview of the arguments.

$\boxed{\textit{Step 1.}}$ To begin with, we cover the vortex set with cylinders of the form $\{B^2_{5C|\varepsilon \ln \varepsilon|}(y_k) \times B^{n-2}_{5C\varepsilon}(x_k)\}_{k=1}^N$ with $x_k = (y_k, z_k) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$ such that $B^{n-2}_{5C\varepsilon}(z_k)$ is a Vitali cover of B_1^{n-2} (see Figure 2.2). Then we name a cylinder *good* if the excess on it is small, and *bad* otherwise. We also define a partition of unity, subordinate to this cover, with derivatives at most $C\varepsilon^{-1}$.

$\boxed{\textit{Step 2.}}$ In each cylinder we pass to the Coulomb gauge, via a function ξ_k with mean equal to the mean of $\theta_h - \theta$ on an appropriate annulus away from the vortex set (note that $\theta - \theta_h$ is well-defined far from the vortex set of both u and u_h) Then we use Gaffney- and Poincaré-type inequalities from Appendix C to derive estimates for $e^{i\xi_k}u - u_h$ and $(\alpha + d\xi_k) - \alpha_h$, where we write

$\nabla = d - i\alpha$. Far from the vorticity set we modify the gauge so that $e^{i\xi_k}u$ and u_h have the same phase. Then we use the exponential decay away from the vortex set, which is where the error ε^β comes from; however, this will be enough to show the classification result in all dimensions, since we are free to take β arbitrarily large.

$\boxed{\textit{Step 3.}}$ We use the estimates on $(\alpha + d\xi_k) - \alpha_h$ (and the mean condition) and Poincaré–Gaffney-type inequalities from Appendix C to bound $\xi_j - \xi_k$ on overlapping cylinders.

$\boxed{\textit{Step 4.}}$ We patch together the ξ_k's with the partition of unity defined in the first step to obtain the function ξ. Then we use the bounds on $\xi_j - \xi_k$ to derive estimates on $(\alpha + d\xi) - \alpha_h$ and $e^{i\xi}u - u_h$.

Proposition 19.2.1 (The interpolation gauge)**.** *For any $\beta > 0$ there exist $\tau_0(n,\beta), \varepsilon_0(n,\beta) > 0$ and $C_0(n,\beta) > 0$ with the following property. Let (u,∇) be a critical pair for E_ε on $\mathbb{R}^n$, with $u(0) = 0$, $\varepsilon \le \varepsilon_0$, and the energy bound*

$$\frac{1}{|B_2^{n-2}|}\int_{B_2^n} e_\varepsilon(u,\nabla) \leqslant 2\pi + \tau_0.$$

Moreover, let $h : B_1^{n-2} \to B_{1/2}^2$ be the Lipschitz approximation defined in Proposition 15.2.1 (for a suitable η chosen later on) and let (u_h, ∇_h) be the pullback pair constructed in Proposition 19.1.1. Then, on a given annulus

$$\mathcal{A}_{s,\delta} := B_1^2 \times (B_{s+\delta}^{n-2} \setminus \overline{B}_s^{n-2})$$

with $\delta \in [C_0\varepsilon, \frac{1}{16}]$ and $s \le \frac{3}{4}$, we can find a gauge transformation

$$(u,\nabla) \mapsto (e^{i\xi}u, \nabla - id\xi),$$

via a smooth function $\xi : \mathcal{A}_{s,\delta} \to \mathbb{R}$ such that:

(i) letting $P : \mathbb{R}^n \to \mathbb{R}^{n-2}$ be the projection onto the last $n-2$ coordinates, we have

$$\begin{aligned}&\int_{\mathcal{A}_{s,\delta}} [\varepsilon^{-2}|e^{i\xi}u - u_h|^2 + |(\alpha + d\xi) - \alpha_h|^2] \\ &\leq C(n,\beta)|\ln\varepsilon|^8 \int_{P(\mathcal{A}_{s-\delta,3\delta})} [\boldsymbol{E}_z + \mathbf{1}_{\mathcal{G}^\eta}\boldsymbol{E}_z|\log \boldsymbol{E}_z|^2] + \varepsilon^\beta;\end{aligned}$$

(ii) letting $Z := \{|u| \leq 3/4\}$ and

$$Z_{C_0\varepsilon|\ln\varepsilon|} := \{x = (y,z) \ : \ \operatorname{dist}(x, Z \cap (B_1^2 \times \{z\})) \leq C_0\varepsilon|\ln\varepsilon|\},$$

the function $e^{i\xi}u \to \mathbb{C} \setminus \{0\}$ has the same phase as u_h far from the vortex set, i.e.,

$$\frac{e^{i\xi}u}{u_h} \in \mathbb{R}^+ \quad \text{on } \mathcal{A}_{s,\delta} \setminus Z_{5C_0|\varepsilon\ln\varepsilon|}.$$

Proof. First we choose ε_0, τ_0 small enough so that Theorem 10.3.5 applies (for $\alpha = 0$). We divide the proof into several steps.

$\boxed{\textit{Covering arguments and a partition of unity.}}$ To begin with, by Theorem 10.3.5, we can find a collection of points

$$\{x_k = (y_k, z_k)\}_{k=1}^N \subset Z \cap \mathcal{A}_{s,\delta}$$

satisfying the following.

(i) The projected collection $\{z_k = P(x_k)\}_{k=1}^N \subset B_{s+\delta}^{n-2} \backslash B_s^{n-2}$ gives a Vitali covering of the projected annulus $P(\mathcal{A}_{s,\delta})$:

$$\begin{aligned}P(\mathcal{A}_{s,\delta}) = B_{s+\delta}^{n-2} \setminus \overline{B}_s^{n-2} &\subseteq \bigcup_{k=1}^N B_{5C_0\varepsilon}^{n-2}(z_k), \\ B_{C_0\varepsilon}^{n-2}(z_j) \cap B_{C_0\varepsilon}^{n-2}(z_k) &= \emptyset \quad \text{for all } j \neq k.\end{aligned} \tag{19.1}$$

This last line shows in particular that $N \leq C(n)\varepsilon^{2-n}$, where by now C_0 depends only on n.

(ii) As a direct consequence of Theorem 10.3.5, we can guarantee that

$$Z_{C_0\varepsilon|\ln\varepsilon|} \cap \mathcal{A}_{s,\delta} \subseteq \bigcup_{k=1}^{N} B^2_{5C_0\varepsilon|\ln\varepsilon|}(y_k) \times B^{n-2}_{5C_0\varepsilon}(z_k). \tag{19.2}$$

(iii) We say that a point $x_k = (y_k, z_k)$ is a *good* point if

$$\sup_{\varepsilon\leqslant r\leqslant 10C_0\varepsilon} r^{2-n}\mathbf{E}_1(u,\nabla, B^2_1 \times B^{n-2}_r(z_k), \mathbb{R}^{n-2}) \leq \eta_0^2,$$

where with a certain abuse of notation we have set

$$\begin{aligned} &\mathbf{E}_1(u,\nabla, B^2_1 \times B^{n-2}_r(z), \mathbb{R}^{n-2}) \\ &:= \int_{B^2_1\times B^{n-2}_r(z)} \left[\sum_{k=3}^{n} |\nabla_{e_k} u|^2 + \varepsilon^2 \sum_{(j,k)\neq(1,2)} \omega(e_j,e_k)^2\right] \end{aligned} \tag{19.3}$$

(note the absence of normalization). Let the set of *good indices* be G. We also denote the set of bad ones by $B := \{1,\ldots,N\} \setminus G$.

(iv) Again as a direct consequence of Theorem 10.3.5 we get that

$$|u_h|, |u| > \frac{3}{4} \quad \text{on} \quad \bigcup_{k=1}^{N} (B^2_{5C_0\varepsilon|\ln\varepsilon|}(y_k) \setminus B^2_{C_0\varepsilon|\ln\varepsilon|}(y_k)) \times B^{n-2}_{5C_0\varepsilon}(z_k). \tag{19.4}$$

Since both u and u_h have degree 1 on each of the previous domains (up to conjugating (u,∇) on $\mathbb{R}^n$), the *difference of phases* $\theta - \theta_h$ is well-defined on these domains.

(v) We also define a partition of unity $\{\phi_k\}_{k=1}^N$, subordinate to the cylinders:

$$\phi_k \in C^1_c(B^2_{5C_0\varepsilon|\ln\varepsilon|}(y_k) \times B^{n-2}_{5C_0\varepsilon}(z_k)), \quad 0 \leq \phi_k \leq 1, \tag{19.5}$$

and

$$\sum_{k=1}^{N} \phi_k = 1 \quad \text{on } Z_{C_0\varepsilon|\ln\varepsilon|} \cap \mathcal{A}_{s,\delta}.$$

We also require that $|d\phi_k| \le \tilde{C}\varepsilon^{-1}$ for all $k = 1, \ldots, N$.

(vi) Up to modifying the ϕ_k's, we can define $\phi_0 \in C_c^1(\mathcal{A}_{s,\delta} \setminus Z_{C_0|\varepsilon\ln\varepsilon|})$ with $0 \le \phi_0 \le 1$ and

$$\begin{aligned} &\phi_0 = 1 \quad \text{on } \mathcal{A}_{s,\delta} \setminus Z_{5C_0|\varepsilon\ln\varepsilon|}, \\ &\phi_0 + \sum_{k=1}^{N} \phi_k = 1 \quad \text{on } \mathcal{A}_{s,\delta}. \end{aligned}$$

Lastly, $|d\phi_0| \le \tilde{C}|\varepsilon\ln\varepsilon|^{-1}$.

Note that $C_0 > 0$ is some large enough constant that we are still free to choose later on ($\tilde{C}$ above depends on C_0). In the sequel we also use the following notation for the excess on each ball:

$$\mathbf{E}(k) := \mathbf{E}(u, \nabla, B_1^2 \times B_{10C_0\varepsilon}^{n-2}(z_k), \mathbb{R}^{n-2}).$$

We define $\mathbf{E}(k) = \mathbf{E}_1(k) + \mathbf{E}_2(k)$ similarly. Since the balls $B_{C_0\varepsilon}^{n-2}(z_k)$ are disjoint, the dilated balls have bounded overlap (i.e., at most $C(n)$ of them intersect), giving

$$\sum_{k=1}^{N} \mathbf{E}(k) \le C \int_{P(\mathcal{A}_{s-\delta,3\delta})} \mathbf{E}_z,$$

and the same holds for $\mathbf{E}_1$ and $\mathbf{E}_2$. With abuse of notation, we also write

$$|\log \mathbf{E}|^2 \mathbf{E}(k) := \begin{cases} \int_{B_{5C_0\varepsilon}^{n-2}(z_k)} |\log \mathbf{E}_z|^2 \mathbf{E}_z & \text{if } k \in G, \\ \int_{B_{5C_0\varepsilon}^{n-2}(z_k)} \mathbf{E}_z & \text{if } k \in B. \end{cases} \tag{19.6}$$

Remark 19.2.2. We will often use the following observation implicitly. For all $z \in B_1^{n-2}$, the results in the previous section show that

$$r^{2-n}\mathbf{E}_1(u, \nabla, B_1^2 \times B_r^{n-2}(z), \mathbb{R}^{n-2}) \leqslant \tilde{\eta}_0^2$$

for an (arbitrarily) small $\tilde{\eta}_0$ and all radii $\Lambda\varepsilon \leqslant r \leqslant \frac{1}{2}$, for some Λ depending on $n, \tilde{\eta}_0$. If $k \in G$ then, for all $z \in B_{5C_0\varepsilon}^{n-2}(z_k)$, we have

$$(C_0\varepsilon)^{2-n}\mathbf{E}_1(u, \nabla, B_1^2 \times B_{C_0\varepsilon}^{n-2}(z), \mathbb{R}^{n-2}) \leqslant 10^{n-2}\eta_0^2.$$

As a consequence of Lemma 14.3.5, if we take $C_0 \geqslant C(n)$ large and η_0^2 very small, we have

$$r^{2-n}\mathbf{E}_1(u, \nabla, B_1^2 \times B_r^{n-2}(z), \mathbb{R}^{n-2}) \leqslant \tilde{\eta}_0^2$$

also for $r \in (0, \Lambda\varepsilon)$, since we have C^1 control on the pair at this scale. We now fix $\tilde{\eta}_0$ such that we actually apply Proposition 15.2.1 for $\eta := \tilde{\eta}_0$, so that *every* $z \in B_{5C_0\varepsilon}^{n-2}(z_k)$ *gives a good slice.*

Gauge fixing on each small cylinder with bounds. Fix $k \in \{1, \dots, n\}$ and consider the unique solution ξ_k to the following Neumann boundary value problem:

$$\begin{cases} \Delta\xi_k = d^*(\alpha - \alpha_h) & \text{in } C_k, \\ \partial_\nu\xi_k = -(\alpha - \alpha_h)(\nu) & \text{at } \partial C_k, \\ \fint_{\mathcal{A}_k}[(\theta + \xi_k) - \theta_h] = 0, \end{cases} \tag{19.7}$$

where

$$C_k := B_{5C_0\varepsilon|\ln\varepsilon|}^2(y_k) \times B_{5C_0\varepsilon}^{n-2}(z_k)$$

and

$$\mathcal{A}_k := (B^2_{5C_0\varepsilon|\ln\varepsilon|}(y_k) \setminus B^2_{C_0\varepsilon|\ln\varepsilon|}(y_k)) \times B^{n-2}_{5C_0\varepsilon}(z_k).$$

Recall that $\theta - \theta_h$ is well-defined on $\mathcal{A}_k$. Then we perform the following gauge transformation in C_k: writing $\nabla = d - i\alpha$, we transform

$$(u, \alpha) \mapsto (e^{i\xi_k}u, \alpha + d\xi_k).$$

Since $d^*[(\alpha + d\xi_k) - \alpha_h] = 0$ and $[(\alpha + d\xi_k) - \alpha_h](\nu) = 0$, we can use the Gaffney–Poincaré-type inequality in Lemma C.0.1:

$$\int_{C_k} |(\alpha + d\xi_k) - \alpha_h|^2 \leq C(n, C_0)|\varepsilon \ln \varepsilon|^2 \int_{C_k} |d\alpha - d\alpha_h|^2. \tag{19.8}$$

We bound separately the contributions of the good set and the bad set, using Lemma 19.2.3 and Lemma 19.2.4 below, obtaining

$$\int_{C_k} |(\alpha + d\xi_k) - \alpha_h|^2 \leq C|\ln\varepsilon|^2|\log \mathbf{E}|^4\mathbf{E}(k) + C\varepsilon^{\beta+3n} \tag{19.9}$$

for some $C = C(n, C_0, \beta) = C(n, \beta)$.

$\boxed{\textit{Gauge fixing far from the vortex set with bounds.}}$ Far from the vortex set, in the set $\mathcal{A}_{s,\delta} \setminus Z_{C_0|\varepsilon\ln\varepsilon|}$, we gauge fix via a function ξ_0 such that $e^{i\xi_0}u/u_h$ is real-valued. Hence, we define

$$\xi_0 := \theta_h - \theta \quad \text{on } \mathcal{A}_{s,\delta} \setminus Z_{C_0|\varepsilon\ln\varepsilon|}; \tag{19.10}$$

note that a priori ξ_0 is well-defined only in the quotient $\mathbb{R}/2\pi\mathbb{Z}$, but this is enough to have a well-defined gauge transformation (in fact, since the vorticity set is included in a $C\varepsilon$-neighborhood of a graph, we can check that $\theta_h - \theta$ is a well-defined real number).

We can estimate $e^{i\xi_0}u - u_h$ and $(\alpha + d\xi_0) - \alpha_h$ in this domain using the exponential decay

(Proposition 13.2.3), as follows:

$$\begin{aligned}
&\int_{\mathcal{A}_{s,\delta}\setminus Z_{C_0|\varepsilon\ln\varepsilon|}} [\varepsilon^{-2}|e^{i\xi_0}u-u_h|^2+|(\alpha+d\xi_0)-\alpha_h|^2] \\
&\leqslant \int_{\mathcal{A}_{s,\delta}\setminus Z_{C_0|\varepsilon\ln\varepsilon|}} [\varepsilon^{-2}||u|-|u_h||^2+2|\alpha-d\theta|^2+2|\alpha_h-d\theta_h|^2] \\
&\leq C(n)\varepsilon^{-2}e^{-K(n)C_0|\ln\varepsilon|}.
\end{aligned}$$

In the last inequality, we used the following observation: since each slice intersects the zero set and $|\ln\varepsilon|\geqslant C(n)$, using Lemma 14.3.5 we see that on $B_1^2\times\{z\}$ the distance from $\{|u|\leqslant 3/4\}$ is comparable with the distance from $(B_1^2\times\{z\})\cap\{|u|\leqslant 3/4\}$.

Taking C_0 large enough, we see that

$$\int_{\mathcal{A}_{s,\delta}\setminus Z_{C_0|\varepsilon\ln\varepsilon|}} [\varepsilon^{-2}|e^{i\xi_0}u-u_h|^2+|(\alpha+d\xi_0)-\alpha_h|^2] \leq \varepsilon^{\beta+3n}. \tag{19.11}$$

$\boxed{\textit{Difference of local gauges in the overlap regions.}}$ Fix $1\leq j<k\leq N$ such that

$$\Omega_{j,k} := C_j\cap C_k \neq \emptyset.$$

Notice that we can bound the L^2 norm of the difference $d\xi_k-d\xi_j$ as follows:

$$\int_{\Omega_{j,k}} |d\xi_j-d\xi_k|^2 \leqslant 2\int_{C_j}|(\alpha+d\xi_j)-\alpha_h|^2+2\int_{C_k}|(\alpha+d\xi_k)-\alpha_h|^2.$$

By (19.9) we then have

$$\int_{\Omega_{j,k}} |d(\xi_j-\xi_k)|^2 \leq C|\ln\varepsilon|^4(|\log\mathbf{E}|^2\mathbf{E}(j)+|\log\mathbf{E}|^2\mathbf{E}(k))+C\varepsilon^{\beta+3n}. \tag{19.12}$$

Our goal is to apply a Poincaré-type inequality on $\Omega_{j,k}$ to estimate $\xi_k-\xi_j$. To this aim, we first look at ξ_j,ξ_k on an appropriate annulus. By the definition of C_j, C_k and the structure of the vortex set in Theorem 10.3.5 we can see that $|x_j - x_k| \leq 20C_0\varepsilon$. We name the midpoint $x_{j,k} = (y_{j,k}, z_{j,k}) := \frac{x_j + x_k}{2}$ and we see that

$$\begin{aligned}\mathcal{A}_{j,k} &:= [B^2_{3C_0\varepsilon|\ln\varepsilon|}(y_{j,k}) \setminus B^2_{2C_0\varepsilon|\ln\varepsilon|}(y_{j,k})] \times [B^{n-2}_{5C_0\varepsilon}(z_j) \cap B^{n-2}_{5C_0\varepsilon}(z_k)] \\ &\subseteq \mathcal{A}_j \cap \mathcal{A}_k,\end{aligned}$$

which is included in $\Omega_{j,k}$. So we can compute that

$$\int_{\mathcal{A}_{j,k}} |\xi_j - \xi_k|^2 \leqslant 2\int_{\mathcal{A}_j} |(\theta + \xi_j) - \theta_h|^2 + 2\int_{\mathcal{A}_k} |(\theta + \xi_k) - \theta_h|^2.$$

We know that $(\theta + \xi_j) - \theta_h$ and $(\theta + \xi_k) - \theta_h$ have zero mean on $\mathcal{A}_j$ and $\mathcal{A}_k$, respectively. Hence, we can apply Lemma C.0.3 on each annulus to see that

$$\int_{\mathcal{A}_{j,k}} \varepsilon^{-2}|\xi_j - \xi_k|^2 \leq C\int_{\mathcal{A}_j} |d(\theta + \xi_j) - d\theta_h|^2 + C\int_{\mathcal{A}_k} |d(\theta + \xi_k) - d\theta_h|^2,$$

and we can bound

$$|d(\theta + \xi_j) - d\theta_h| \leqslant |\alpha - d\theta| + |\alpha_h - d\theta_h| + |\alpha + d\xi_j - \alpha_h|.$$

As before, on $\mathcal{A}_j$ we have

$$|\alpha - d\theta|^2 + |\alpha_h - d\theta_h|^2 \leqslant |u|^{-2}|\nabla u|^2 + |u_h|^{-2}|\nabla_h u_h|^2 \leqslant \varepsilon^{\beta+3n}$$

by exponential decay, and the same holds for k. Together with (19.9) we thus estimate

$$\int_{\mathcal{A}_{j,k}} \varepsilon^{-2}|\xi_j - \xi_k|^2 \leq C|\ln\varepsilon|^4(|\log \mathbf{E}|^2\mathbf{E}(j) + |\log \mathbf{E}|^2\mathbf{E}(k)) + C\varepsilon^{\beta+3n}. \tag{19.13}$$

By (19.12)–(19.13), using Lemma C.0.3 and Remark C.0.4, we arrive at

$$\int_{\Omega_{j,k}} \varepsilon^{-2}|\xi_j - \xi_k|^2 \le C|\ln\varepsilon|^4(|\log \mathbf{E}|^2\mathbf{E}(j) + |\log \mathbf{E}|^2\mathbf{E}(k)) + \varepsilon^{\beta+3n}. \tag{19.14}$$

We also need to estimate the difference of $\xi_k - \xi_0$ for all $1 \le k \le N$. Defining

$$\Omega_{0,k} := C_k \cap [\mathcal{A}_{s,\delta} \setminus Z_{C_0|\varepsilon\ln\varepsilon|}],$$

we see that

$$\Omega_{0,k} \subseteq \mathcal{A}'_k := [B^2_{5C_0\varepsilon|\ln\varepsilon|}(y_k) \setminus B^2_{(C_0/2)\varepsilon|\ln\varepsilon|}(y_k)] \times B^{n-2}_{5C_0\varepsilon}(z_k).$$

Note that by (19.10) we have $\xi_0 = \theta_h - \theta$ in $\Omega_{0,k}$. Hence, we can apply Lemma C.0.3 and compute that

$$\begin{aligned}
\varepsilon^{-2}\int_{\Omega_{0,k}} |\xi_k - \xi_0|^2 &\le \varepsilon^{-2}\int_{\mathcal{A}'_k} |(\theta + \xi_k) - \theta_h|^2 \\
&\le \int_{\mathcal{A}'_k} |d(\theta + \xi_k) - d\theta_h|^2 \\
&\le \int_{\mathcal{A}'_k} [|\alpha - d\theta|^2 + |\alpha_h - d\theta_h|^2 + |(\alpha + d\xi_k) - \alpha_h|^2],
\end{aligned}$$

where again we used the fact that $(\theta+\xi_k)-\theta_h$ has zero mean on $\mathcal{A}_k \subset \mathcal{A}'_k$. Summing the previous bounds and noting that there are at most $(C\varepsilon^{2-n})^2$ pairs (j,k), while any point belongs to at most $C = C(n, C_0)$ domains $\Omega_{j,k}$, we arrive at

$$\begin{aligned}
&\sum_{0\le j<k\le N} \int_{\Omega_{j,k}} \varepsilon^{-2}|\xi_j - \xi_k|^2 \\
&\le C|\ln\varepsilon|^4 \int_{P(\mathcal{A}_{s-\delta,3\delta})} |\log \mathbf{E}_z|^2\mathbf{E}_z + C\mathbf{E} + C\varepsilon^{\beta+1},
\end{aligned} \tag{19.15}$$

for some $C = C(n, \beta)$ (recall (19.6)).

$\boxed{\textit{Constructing the global gauge via the partition of unity.}}$ Recall the definition of the partition of unity in (19.5). We define the global gauge transformation function as follows:

$$\xi := \sum_{k=0}^{N} \phi_k \xi_k \quad \text{on } \mathcal{A}_{s,\delta}. \tag{19.16}$$

Then we estimate

$$\begin{aligned}
&\int_{\mathcal{A}_{s,\delta}} \varepsilon^{-2} \left| e^{i\xi} u - \sum_{k=0}^{N} \phi_k e^{i\xi_k} u \right|^2 \\
&\leq \varepsilon^{-2} \int_{\mathcal{A}_{s,\delta}} \sum_{k=0}^{N} \phi_k |\xi - \xi_k|^2 \\
&\leq 2 \sum_{j<k} \int_{\Omega_{j,k}} \varepsilon^{-2} |\xi_j - \xi_k|^2 \\
&\leq C |\ln \varepsilon|^4 \int_{P(\mathcal{A}_{s-\delta,3\delta})} |\log \mathbf{E}_z|^2 \mathbf{E}_z + C\mathbf{E} + C\varepsilon^{\beta+1}.
\end{aligned}$$

In particular,

$$\begin{aligned}
&\int_{\mathcal{A}_{s,\delta}} \varepsilon^{-2} |e^{i\xi} u - u_h|^2 \\
&\leq 2 \int_{\mathcal{A}_{s,\delta}} \left[\varepsilon^{-2} \left| e^{i\xi} u - \sum_{k=0}^{N} \phi_k e^{i\xi_k} u \right|^2 + \sum_{k=0}^{N} \phi_k |e^{i\xi_k} u - u_h|^2 \right] \\
&\leq C |\ln \varepsilon|^8 \int_{P(\mathcal{A}_{s-\delta,3\delta})} |\log \mathbf{E}_z|^2 \mathbf{E}_z + C\mathbf{E} + C\varepsilon^{\beta+1},
\end{aligned} \tag{19.17}$$

where we used Lemma 19.2.3 and Lemma 19.2.4 to estimate the term involving $e^{i\xi_k} u - u_h$.

Moreover, for the connection part, we can bound

$$\begin{aligned}
&\int_{\mathcal{A}_{s,\delta}} |(\alpha + d\xi) - \alpha_h|^2 \\
&\leq 2 \int_{\mathcal{A}_{s,\delta}} \left[\sum_{k=0}^{N} \phi_k |(\alpha + d\xi_k) - \alpha_h|^2 + \left| \sum_{k=0}^{N} d\phi_k \xi_k \right|^2 \right].
\end{aligned} \tag{19.18}$$

The first term is bounded by (19.9) and (19.11). We are left to bound the last term. Since the functions ϕ_k form a partition of unity, we have

$$\sum_{k=0}^{N} d\phi_k(z) = 0.$$

We can then write

$$\sum_{k=0}^{N} d\phi_k \xi_k = \sum_{j,k=0}^{N} \phi_j d\phi_k (\xi_k - \xi_j).$$

Since $|d\phi_k| \leqslant C\varepsilon^{-1}$, the last term above is bounded by

$$C\varepsilon^{-2} \sum_{j<k} \int_{\Omega_{j,k}} |\xi_j - \xi_k|^2,$$

which is a quantity that we already estimated. Combining these bounds, we see that

$$\begin{aligned}
&\int_{\mathcal{A}_{s,\delta}} [\varepsilon^{-2} |e^{i\xi} u - u_h|^2 + |(\alpha + d\xi) - \alpha_h|^2] \\
&\leq C |\ln \varepsilon|^8 \int_{P(\mathcal{A}_{s-\delta,3\delta})} [\mathbf{E}_z + \mathbf{1}_{\mathcal{G}^\eta} \mathbf{E}_z |\log \mathbf{E}_z|^2] + \varepsilon^\beta.
\end{aligned}$$

This is indeed the desired conclusion. □

We now turn to the bounds which were postponed in the previous proof.

Lemma 19.2.3. *Assume that $k \in G$. Then*

$$\int_{C_k} \varepsilon^{-2} |e^{i\xi_k} u - u_h|^2 \le C |\ln \varepsilon|^8 |\log \boldsymbol{E}|^2 \boldsymbol{E}(k) + C\varepsilon^{\beta+3n}$$

and

$$\int_{C_k} \varepsilon^2 |d\alpha - d\alpha_h|^2 \le C |\ln \varepsilon|^4 \eta^{-2} |\log \boldsymbol{E}|^2 \boldsymbol{E}(k) + C\varepsilon^{\beta+3n}.$$

Proof. We bound each part separately.

$\boxed{\textit{Estimating } d\alpha - d\alpha_h.}$ Recalling the definition of $\mathbf{E}_1$ in (19.3) and the construction of α_h, we have

$$\int_{C_k} \varepsilon^2 |d\alpha - d\alpha_h|^2 \leqslant \int_{C_k} \varepsilon^2 |d\alpha(e_1, e_2) - d\alpha_h(e_1, e_2)|^2 + C\mathbf{E}_1(k) + C \int_{B^{n-2}_{5C_0\varepsilon}(z_k)} |dh|^2.$$

We are going to use some estimates from [45] which are slightly more refined than (13.3). Compared to the main result of [45], these hold under some additional assumptions, which are however satisfied on good slices: in particular, for any $z \in B^{n-2}_{5C_0\varepsilon}(z_k)$, the function u vanishes linearly at a unique point along the slice $B^2_1 \times \{z\}$. We will often compare u with the function u_{h_0}, where h_0 is the function built in Proposition 15.3.1, whose graph approximates the zero set; along the good slice, this function vanishes at the same point as u, and is just a translation of the standard degree-one planar solution.

Specifically, using an ε-rescaling of (13.3) and Theorem 13.4.2 (applied with $N = 1$), we have the following estimate:

$$\begin{aligned}\int_{B_1^2\times\{z\}} & [\varepsilon^{-2}||u| - |u_{h_0}||^2 + |u_{h_0}|^{2+1/2}|(\alpha - d\theta)_{(1,2)} - (\alpha_{h_0} - d\theta_{h_0})_{(1,2)}|^2 \\ & + \varepsilon^2|d\alpha(e_1, e_2) - d\alpha_{h_0}(e_1, e_2)|^2] \\ \leq C\mathbf{E}_z, &\end{aligned} \tag{19.19}$$

for an absolute constant C, where the subscript $(1,2)$ means that we restrict the one-form along the slice.

Now by the construction in Proposition 19.1.1 we can see that (u_h, α_h), along the slice $B_1^2 \times \{z\}$, is equal to (u_{h_0}, α_{h_0}) translated to vanish at the barycenter $\Phi_{\chi(x_1,x_2)}(z)$. As shown in Lemma B.0.1, the translation is by a vector v with $|v| \leq C\varepsilon|\log \mathbf{E}_z|\sqrt{\mathbf{E}_z} + e^{-K/\varepsilon}$. By the mean value theorem, we then have

$$\begin{aligned}\int_{B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k)\times\{z\}} & [\varepsilon^2|d\alpha_h(e_1, e_2) - d\alpha_{h_0}(e_1, e_2)|^2 + |(\alpha_h - \alpha_{h_0})_{(1,2)}|^2 \\ & + \varepsilon^{-2}|u_h - u_{h_0}|^2] \\ \leqslant C\varepsilon^2|\log\varepsilon|^2 \cdot |v|^2 \cdot C\varepsilon^{-4} & \\ \leqslant C|\ln\varepsilon|^2|\log \mathbf{E}_z|^2\mathbf{E}_z + e^{-K/\varepsilon}, &\end{aligned} \tag{19.20}$$

since $\mathbf{E}_z$ is bounded on good slices and the differential of each quantity (such as $\varepsilon d\alpha_h(e_1, e_2)$ and so on) is bounded by $C\varepsilon^{-2}$. The claimed estimate follows by combining the previous bounds (together with item (iv) from Proposition 15.2.1, which gives $|dh|^2(z) \leqslant C\mathbf{E}_z + e^{-K/\varepsilon}$).

$\boxed{\textit{Estimating } e^{i\xi_k}u - u_h.}$ Writing formally $u = |u|e^{i\theta}$ and using a similar notation for u_h and u_{h_0}, recall that on the annulus

$$\mathcal{A}_{k,z} := [B^2_{5C_0\varepsilon|\ln\varepsilon|}(y_k) \setminus B^2_{C_0\varepsilon|\ln\varepsilon|}(y_k)] \times \{z\}$$

the differences $\theta - \theta_h$, $\theta - \theta_{h_0}$ and $\theta_h - \theta_{h_0}$ are well-defined. We record the following estimate:

$$\int_{\mathcal{A}_{k,z}} \varepsilon^{-2} |\theta_h - \theta_{h_z}|^2 \leq C |\log \mathbf{E}_z|^2 \mathbf{E}_z. \tag{19.21}$$

This holds again by the mean value theorem, since $|(d\theta_h)_{(1,2)}|(y) \leqslant C|y - y_k|^{-1}$. We are going to use the Caffarelli–Kohn–Nirenberg-type inequality from Lemma C.0.2, which implies that

$$\int_{B_R^2(y_k)} |y - h_0(z)|^2 |f(y)|^2 \leq CR^{3/2} \int_{B_R^2(y_k)} |y - h_0(z)|^{2+1/2} |df(y)|^2,$$

for $f \in C_c^1(B_R^2(y_k))$, with $R := 5C_0\varepsilon|\log\varepsilon|$ (since there exists a biLipschitz transformation sending $B_R^2(y_k)$ to itself and mapping the origin to $h_0(z)$). Recalling that the standard degree-one solution vanishes linearly at the origin, by the construction of u_{h_0} in Proposition 19.1.1 we have

$$C^{-1} \leq \frac{|u_{h_0}|(y,z)}{\min\{\varepsilon^{-1}|y - h_0(z)|, 1\}} \leq C$$

on the good slice, for some universal constant C. Moreover,

$$1 \leq \frac{\varepsilon^{-1}|y - h_0(z)|}{\min\{\varepsilon^{-1}|y - h_0(z)|, 1\}} \leq C|\ln\varepsilon| \quad \text{for all } y \in B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k).$$

Hence, given a C^1 function f on $B^2_{5C_0\varepsilon|\ln\varepsilon|}(y_k) \times \{z\}$ vanishing near the boundary, we can write

$$\int_{B^2_{5C_0\varepsilon|\ln\varepsilon|}(y_k)\times\{z\}} |u_{h_0}|^2 |f|^2 \leq C\varepsilon^2 |\ln\varepsilon|^4 \int_{B^2_{5C_0\varepsilon|\ln\varepsilon|}(y_k)\times\{z\}} |u_{h_0}|^{2+1/2} |df|^2. \tag{19.22}$$

To estimate $ue^{i\xi_k} - u_{h_0}$, we first notice that u and u_{h_0} have the same unique zero point (with the same degree around it), and hence the difference $\theta - \theta_{h_0}$ gives a well-defined smooth function on the full slice.

We define a cut-off $\chi : B_1^2 \to \mathbb{R}$ with $\chi = 1$ on $B^2_{C_0\varepsilon|\ln\varepsilon|}(y_k)$ and $\chi = 0$ outside of $B^2_{5C_0\varepsilon|\ln\varepsilon|}(y_k)$, with $|d\chi| \le C|\varepsilon \ln \varepsilon|^{-1}$. Then we use the first term of (19.19) to bound $|u| - |u_{h_z}|$ and (19.22) to see that

$$\begin{aligned}
&\varepsilon^{-2} \int_{B^2_{5C_0\varepsilon|\ln\varepsilon|}(y_k)\times\{z\}} \chi^2 |e^{i\xi_k} u - u_{h_0}|^2 \\
&\le C \int_{B^2_{5C_0\varepsilon|\ln\varepsilon|}(y_k)\times\{z\}} \chi^2 \left[\frac{\big||u| - |u_{h_0}|\big|^2}{\varepsilon^2} + \frac{|u_{h_0}|^2}{\varepsilon^2} |(\theta + \xi_k) - \theta_{h_0}|^2 \right] \\
&\le C\mathbf{E}_z + C(\mathrm{I} + \mathrm{II}),
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I} &:= |\ln\varepsilon|^4 \int_{B^2_{5C_0\varepsilon|\ln\varepsilon|}(y_k)\times\{z\}} |u_{h_0}|^{2+1/2} \chi^2 |d(\theta + \xi_k - \theta_{h_0})_{(1,2)}|^2, \\
\mathrm{II} &:= |\ln\varepsilon|^4 \int_{B^2_{5C_0\varepsilon|\ln\varepsilon|}(y_k)\times\{z\}} |u_{h_0}|^{2+1/2} |d\chi|^2 |\theta + \xi_k - \theta_{h_0}|^2.
\end{aligned}$$

First we estimate **I** using the second term in (19.19) and (19.20) (to replace α_{h_0} with α_h):

$$\mathbf{I} \le C|\ln\varepsilon|^6 |\log \mathbf{E}_z|^2 \mathbf{E}_z + C|\ln\varepsilon|^4 \int_{B^2_{5C_0\varepsilon|\ln\varepsilon|}(y_k)\times\{z\}} \chi^2 |(\alpha + d\xi_k) - \alpha_h|^2.$$

Then we estimate **II**: we note that $|d\chi|$ is supported in $\mathcal{A}_{k,z}$ and $|d\chi| \le C|\varepsilon \ln\varepsilon|^{-1}$, and hence we can use (19.21) to estimate

$$\mathbf{II} \le C|\ln\varepsilon|^2 |\log \mathbf{E}_z|^2 \mathbf{E}_z + C|\ln\varepsilon|^2 \int_{\mathcal{A}_{k,z}} \varepsilon^{-2} |\theta + \xi_k - \theta_h|^2.$$

Putting **I** and **II** together and integrating over $B^{n-2}_{5C_0\varepsilon}(z_k)$, we see that

$$\begin{aligned}
&\int_{C_k} \varepsilon^{-2}|e^{i\xi_k}u - u_{h_0}|^2 \\
&\leq C|\ln\varepsilon|^6|\log\mathbf{E}|^2\mathbf{E}(k) + C|\ln\varepsilon|^4\int_{C_k}|\alpha + d\xi_k - \alpha_h|^2 \\
&\quad + C|\ln\varepsilon|^2\int_{\mathcal{A}_k}\varepsilon^{-2}|\theta + \xi_k - \theta_h|^2 \\
&\leq C|\ln\varepsilon|^8|\log\mathbf{E}|^2\mathbf{E}(k) + C|\ln\varepsilon|^2\int_{\mathcal{A}_k}\varepsilon^{-2}|\theta + \xi_k - \theta_h|^2 + \varepsilon^{\beta+3n},
\end{aligned}$$

where we used (19.9) (which uses only the previous bound on $d\alpha - d\alpha_h$). Recalling that we imposed

$$\int_{\mathcal{A}_k}(\theta + \xi_k - \theta_h) = 0,$$

we can apply Lemma C.0.3 (suitably rescaled) and (19.9) another time to see that

$$\begin{aligned}
&|\ln\varepsilon|^2\int_{\mathcal{A}_k}\varepsilon^{-2}|\theta + \xi_k - \theta_h|^2 \\
&\leq C|\ln\varepsilon|^2\int_{\mathcal{A}_k}|d(\theta + \xi_k) - d\theta_h|^2 \\
&\leq C|\ln\varepsilon|^2\int_{\mathcal{A}_k}[|\alpha - d\theta|^2 + |\alpha_h - d\theta_h|^2 + |(\alpha + d\xi_k) - \alpha_h|^2] \\
&\leq C|\ln\varepsilon|^6|\log\mathbf{E}|^2\mathbf{E}(k) + \varepsilon^{\beta+3n}
\end{aligned} \tag{19.23}$$

up to taking C_0 large enough (the last inequality follows from the exponential decay of energy); combining these bounds with (19.20), we get the desired bound for $e^{i\xi_k}u - u_h$. □

Lemma 19.2.4. *For $k \in B$ we have*

$$\int_{C_k}[\varepsilon^{-2}|e^{i\xi_k}u - u_h|^2 + \varepsilon^2|d\alpha - d\alpha_h|^2] \leq C|\ln\varepsilon|^2\boldsymbol{E}_1(k).$$

Proof. On the bad set we simply use L^∞ bounds: we have

$$\begin{aligned}
&\int_{C_k} [\varepsilon^{-2}|e^{i\xi_k}u - u_h|^2 + \varepsilon^2|d\alpha - d\alpha_h|^2] \\
&\leq C|\ln\varepsilon|^2\varepsilon^{n-2} \\
&\leq C|\ln\varepsilon|^2\mathbf{E}_1(k),
\end{aligned} \tag{19.24}$$

where we used the definition of bad index in the last inequality. □

Corollary 19.2.5. *We have*

$$\begin{aligned}
&\int_{\mathcal{A}_{s,\delta}} [\varepsilon^{-2}|e^{i\xi}u - u_h|^2 + |(\alpha + d\xi) - \alpha_h|^2] \\
&\leq C(n,\beta)|\ln\varepsilon|^{10}\int_{P(\mathcal{A}_{s-\delta,3\delta})} \boldsymbol{E}_z + \varepsilon^\beta.
\end{aligned}$$

Proof. Recalling that for a good z the sliced excess $\mathbf{E}_z$ is small, it suffices to split the integral of $\mathbf{E}_z|\log\mathbf{E}_z|^2$ on the two sets $\{\mathbf{E}_z \leqslant \varepsilon^{\beta+1}\}$ and $\{\mathbf{E}_z > \varepsilon^{\beta+1}\}$. On the second one, we bound $|\log\mathbf{E}_z|^2 \leqslant C|\log\varepsilon|^2$, while on the first one we have $\mathbf{E}_z|\log\mathbf{E}_z|^2 \leqslant C\varepsilon^{\beta+1}|\log\varepsilon|^2$. □

Chapter 20

Proof of a stronger decay for local minimizers

20.1 Strong approximation of the excess for minimizers

In this section we use variational arguments and the estimates from Proposition 19.2.1 to construct competitors. As a consequence, we prove that the full excess **E** is well approximated by the Dirichlet energy of a harmonic approximation w built as in Proposition 16.1.2.

Proposition 20.1.1 (Strong harmonic approximation of minimizers)**.** *For any $\nu, \beta > 0$ and any radius $0 < s < 1$ there exist three small constants $\varepsilon_0(n, s, \nu, \beta), \tau_0(n, s, \nu, \beta), \eta_0(n, \nu, \beta) > 0$ with the following properties. Let (u, ∇) be a minimizer of E_ε defined on $B_2^n(0)$, with $\varepsilon \leq \varepsilon_0$ and the energy bound*

$$\frac{1}{|B_2^{n-2}|}\int_{B_2^n} e_\varepsilon(u, \nabla) \leqslant 2\pi + \tau_0.$$

After a suitable rotation, let $h : B_1^{n-2} \to B_1^2$ be the Lipschitz approximation defined in Proposition 15.2.1 with $\eta := \eta_0$. Then the following holds, assuming

$$Ce^{-K/\varepsilon} \leqslant C\varepsilon^\beta \leqslant \boldsymbol{E} := \boldsymbol{E}(u, \nabla, B_2^n, \mathbb{R}^{n-2})$$

for some $C = C(n, \nu, \beta)$ and $K = K(n)$: there exists a harmonic function $w : B_1^{n-2} \to \mathbb{R}^2$ such that

(i) $\int_{B_1^{n-2}} |dw|^2 \leqslant C(n)$;

(ii) *we have*

$$\int_{B_1^{n-2}} \left| \frac{h - (h)_{B_1^{n-2}}}{\sqrt{\boldsymbol{E}}} - w \right|^2 \leqslant \nu,$$

where $(h)_{B_1^{n-2}}$ *is the average;*

(iii) *most importantly, we have*

$$\int_{B_s^{n-2}} \boldsymbol{E}_z \leqslant \boldsymbol{E} \int_{B_s^{n-2}} \frac{|dw|^2}{2} + \nu \boldsymbol{E}.$$

Proof. We prove the statement by compactness and contradiction. Fix ν, β, s and assume that there exist sequences $\varepsilon_k, \tau_k \to 0$ and a sequence of minimizers (u_k, ∇_k) for E_{ε_k} with the previous energy bound for $\tau_0 = \tau_k$, violating the conclusion. Moreover, let $h_k : B_1^{n-2} \to B_1^2$ be the Lipschitz approximation for the threshold η_0, to be chosen below.

Lower bound on the energy of the given pair. First of all, recalling item (i) in Proposition 15.2.1 we have

$$\int_{B_1^{n-2}} |dh_k|^2 \leq C(n)\mathbf{E}^{(k)}, \quad \mathbf{E}^{(k)} := \mathbf{E}(u_k, \nabla_k, B_2^n, \mathbb{R}^{n-2}),$$

for k large enough. Hence, up to a subsequence, we can extract a weak limit

$$\frac{h_k - (h_k)_{B_1^{n-2}}}{\sqrt{\mathbf{E}^{(k)}}} \rightharpoonup w$$

in $W^{1,2}$, so that

$$\int_{B_1^{n-2}} |dw|^2 \leqslant C(n).$$

By Lemma 16.1.1 and Proposition 16.1.2, w is harmonic with $w(0) = 0$. This shows that the first two conclusions hold, so we must have

$$\int_{B_s^{n-2}} \mathbf{E}_z^{(k)} > \mathbf{E}^{(k)} \int_{B_s^{n-2}} \frac{|dw|^2}{2} + \nu \mathbf{E}^{(k)}. \tag{20.1}$$

By Lemma 14.3.6 and the bound $Ce^{-K/\varepsilon} \leqslant \frac{\nu}{5}\mathbf{E}^{(k)}$, this gives

$$\frac{1}{2\pi} \int_{B_1^2 \times B_s^{n-2}} e_{\varepsilon_k}(u_k, \nabla_k) > |B_s^{n-2}| + \mathbf{E}^{(k)} \int_{B_s^{n-2}} \frac{|dw|^2}{2} + \frac{4\nu}{5}\mathbf{E}^{(k)}.$$

Let $a, b \in (s, 1)$ with $a < b$, which we write as $b = a + 4\delta$. Calling $\mathcal{G}^k$ the good set for (u_k, ∇_k), since the indicator function $\mathbf{1}_{(B_a^{n-2} \setminus B_s^{n-2}) \cap \mathcal{G}^k} \to 1$ strongly $L^2(B_a^{n-2} \setminus B_s^{n-2})$, we have

$$\mathbf{1}_{(B_a^{n-2} \setminus B_s^{n-2}) \cap \mathcal{G}^k} \frac{dh_k}{\sqrt{\mathbf{E}^{(k)}}} \rightharpoonup dw$$

weakly in this space, and hence

$$\int_{B_a^{n-2} \setminus B_s^{n-2}} \frac{|dw|^2}{2} \leqslant \liminf_{k \to \infty} \int_{(B_a^{n-2} \setminus B_s^{n-2}) \cap \mathcal{G}^k} \frac{|dh_k|^2}{2\mathbf{E}^{(k)}}.$$

Using item (iv) in Proposition 15.2.1 and the assumption $\mathbf{E}^{(k)} \geqslant C\varepsilon_k^\beta \geqslant Ce^{-K/\varepsilon_k}$, we deduce

$$\int_{B_a^{n-2} \setminus B_s^{n-2}} \frac{|dw|^2}{2} \leqslant \liminf_{k \to \infty} \int_{(B_a^{n-2} \setminus B_s^{n-2}) \cap \mathcal{G}^k} \frac{\mathbf{E}_z^{(k)}}{\mathbf{E}^{(k)}}.$$

Combined with (20.1), this gives

$$\int_{B_a^{n-2}} \mathbf{E}_z^{(k)} > \mathbf{E}^{(k)} \int_{B_a^{n-2}} \frac{|dw|^2}{2} + \frac{3\nu}{4}\mathbf{E}^{(k)}.$$

Using again Lemma 14.3.6 and $\mathbf{E}^{(k)} \geqslant C\varepsilon_k^\beta \geqslant Ce^{-K/\varepsilon_k}$, we obtain

$$\frac{1}{2\pi}\int_{B_1^2\times B_a^{n-2}} e_{\varepsilon_k}(u_k,\nabla_k) > |B_a^{n-2}| + \mathbf{E}^{(k)}\int_{B_a^{n-2}}\frac{|dw|^2}{2} + \frac{3\nu}{5}\mathbf{E}^{(k)}. \tag{20.2}$$

Note that, for a fixed small $\delta > 0$ to be specified later, we can find a and $b = a + 4\delta$ in $(s, 1)$ such that

$$\int_{B_b^{n-2}\setminus B_a^{n-2}} [|dh_k|^2 + \mathbf{E}^{(k)}|dw|^2 + \mathbf{E}_z^{(k)}] \leqslant C(n,s)\delta\mathbf{E}^{(k)}, \tag{20.3}$$

along a subsequence, by the classical pigeonhole argument.

Now we take a cut-off function χ such that $\chi = 1$ on B_a^{n-2} and $\chi = 0$ outside of $B_{a+\delta}^{n-2}$, and we let

$$f_k := (1-\chi)h_k + \chi(\sqrt{\mathbf{E}^{(k)}}w + (h_k)_{B_1^{n-2}}).$$

Since $\|h_k - (h_k)_{B_1^{n-2}} - \sqrt{\mathbf{E}^{(k)}}w\|_{L^2}^2 = o(\mathbf{E}^{(k)})$, the Dirichlet energy of f_k on $B_b^{n-2}\setminus B_a^{n-2}$ is

$$\int_{B_b^{n-2}\setminus B_a^{n-2}} \left[(1-\chi)^2\frac{|dh_k|^2}{2} + \mathbf{E}^{(k)}(2\chi-\chi^2)\frac{|dw|^2}{2}\right] + o(\mathbf{E}^{(k)}).$$

In particular, by (20.3) we have

$$\int_{B_b^{n-2}\setminus B_a^{n-2}} |df_k|^2 \leqslant C\delta\mathbf{E}^{(k)}. \tag{20.4}$$

We apply Proposition 19.1.1 to obtain a new pair (u_{f_k}, ∇_{f_k}).

$\boxed{\textit{Construction of the competitor.}}$ We want to glue the latter to (u_k, ∇_k) in a suitable annular region and obtain a new pair whose energy in $B_1^2\times B_b^{n-2}$ is strictly lower than (u_k,∇_k), obtaining a contradiction to minimality. From now on, we restrict attention to the region $B_1^2\times B_b^{n-2}$. We will also drop the subscript k in the sequel. Note that $f = h$ on $B_{a+4\delta}^{n-2}\setminus B_{a+\delta}^{n-2}$.

For technical reasons, it will be convenient to glue on an annulus of width $\sqrt{\varepsilon}$. We first select

$t \in [a+2\delta, a+3\delta]$ such that

$$\int_{B^{n-2}_{t+2\sqrt{\varepsilon}} \setminus B^{n-2}_{t-\sqrt{\varepsilon}}} [\mathbf{E}_z + |dh|^2] \leqslant C(n,s,\delta)\sqrt{\varepsilon}\mathbf{E}. \tag{20.5}$$

We first apply Proposition 19.2.1 and Corollary 19.2.5 to replace (u,∇) with a gauge-equivalent pair, still denoted (u,∇), such that $\frac{u}{|u|} = \frac{u_f}{|u_f|}$ on $(B_1^2 \setminus B_{1/2}^2) \times B_b^{n-2}$, with

$$\int_{\mathcal{A}} [\varepsilon^{-2}|u-u_h|^2 + |\alpha - \alpha_h|^2] \leqslant C|\ln\varepsilon|^{10} \int_{P(\hat{\mathcal{A}})} \mathbf{E}_z + \varepsilon^{\beta},$$

where $\mathcal{A} := B_1^2 \times (B^{n-2}_{t+\sqrt{\varepsilon}} \setminus B^{n-2}_t)$ and $\hat{\mathcal{A}} := B_1^2 \times (B^{n-2}_{t+2\sqrt{\varepsilon}} \setminus B^{n-2}_{t-\sqrt{\varepsilon}})$. In particular, by (20.5) we have

$$\int_{\mathcal{A}} [\varepsilon^{-2}|u-u_h|^2 + |\alpha - \alpha_h|^2] \leqslant C\sqrt{\varepsilon}|\ln\varepsilon|^{10} \int_{P(\hat{\mathcal{A}})} \mathbf{E}_z + \varepsilon^{\beta} = o(\mathbf{E}) + \varepsilon^{\beta}, \tag{20.6}$$

where the notation $o(\mathbf{E}) = o(\mathbf{E}^{(k)})$ indicates a quantity infinitesimal with respect to $\mathbf{E}^{(k)}$, as $k \to \infty$.

We take another cut-off function φ with $\varphi = 1$ on $B_1^2 \times B_t^{n-2}$ and $\varphi = 0$ outside of $B_1^2 \times B^{n-2}_{t+\sqrt{\varepsilon}}$. On $B_1^2 \times B_b^{n-2}$, we define

$$\tilde{u} := (1-\varphi)u + \varphi u_h, \quad \tilde{\alpha} := (1-\varphi)\alpha + \varphi\alpha_h.$$

We claim that

$$\int_{P(\mathcal{A})} \tilde{\mathbf{E}}_z \leqslant o(\mathbf{E}) + C\varepsilon^{\beta}. \tag{20.7}$$

Once this is done, using Lemma 14.3.6, we obtain

$$\frac{1}{2\pi}\int_{P(\mathcal{A})} e_\varepsilon(\tilde{u},\tilde{\nabla}) \leqslant |P(\mathcal{A})| + o(\mathbf{E}_z) + C\varepsilon^{\beta},$$

and hence by Proposition 19.1.1, together with (20.3) and (20.4), we get

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{B_1^2\times B_b^{n-2}} e_\varepsilon(\tilde{u},\tilde{\nabla})\\
&\leqslant |B_b^{n-2}| + (1+O(\eta_0^2))\int_{B_b^{n-2}}\frac{|df|^2}{2} + \int_{B_b^{n-2}\setminus B_a^{n-2}} \mathbf{E}_z + o(\mathbf{E}) + C\varepsilon^\beta\\
&\leqslant |B_b^{n-2}| + \mathbf{E}\int_{B_a^{n-2}}\frac{|dw|^2}{2} + \int_{B_b^{n-2}\setminus B_a^{n-2}}[|df|^2+\mathbf{E}_z] + C\eta_0^2\mathbf{E} + o(\mathbf{E}) + C\varepsilon^\beta\\
&\leqslant |B_b^{n-2}| + \mathbf{E}\int_{B_a^{n-2}}\frac{|dw|^2}{2} + \frac{\nu}{5}\mathbf{E},
\end{aligned}
$$

once we take η_0 and δ small enough. In the same way, (20.2) gives

$$
\frac{1}{2\pi}\int_{B_1^2\times B_b^{n-2}} e_\varepsilon(u,\nabla) > |B_b^{n-2}| + \mathbf{E}\int_{B_a^{n-2}}\frac{|dw|^2}{2} + \frac{2\nu}{5}\mathbf{E}.
$$

This gives a contradiction: near $\partial B_1^2 \times B_{t+\sqrt{\varepsilon}}^{n-2}$, using the fact that $\tilde{u}$ and u_f have the same phase, it is easy to modify $(\tilde{u},\tilde{\nabla})$ in order to make it agree with (u,∇) (while this already holds on $\partial B_1^2 \times (B_b^{n-2} \setminus B_{t+\sqrt{\varepsilon}}^{n-2})$), in a way which changes the energy by $O(e^{-K/\varepsilon}) \leqslant \frac{\nu}{5}\mathbf{E}$: it is enough to interpolate between the two pairs on the set

$$
(B_1^2 \setminus B_{1/2}^2) \times B_b^{n-2},
$$

using the fact that here the energy density is exponentially small, and hence we can write $|\tilde{u} - u_f| = |(1-|\tilde{u}|) - (1-|u_f|)| \leqslant e^{-K/\varepsilon}$ and $|\tilde{\alpha} - \alpha_f| \leqslant e^{-K/\varepsilon}$ (since, writing $\tilde{u} = e^{i\tilde{\theta}}$, we have $|d\tilde{\theta} - \tilde{\alpha}| \leqslant |\tilde{u}|^{-1}|\tilde{\nabla}\tilde{u}|$ and similarly $|d\tilde{\theta} - \alpha_f| = |d\theta_f - \alpha_f| \leqslant |u_f|^{-1}|\nabla_f u_f|$).

Bounding the energy on the interpolation annulus. It remains to check the previous claim. We first write

$$
\begin{aligned}
2\pi\tilde{\mathbf{E}}_z = \int_{B_1^2\times\{z\}} \sum_{j\geqslant 3} |\tilde{\nabla}_{e_j}\tilde{u}|^2 + \sum_{(j,j')\neq(1,2)} \varepsilon^2|d\tilde{\alpha}(e_j,e_{j'})|^2 + |\tilde{\nabla}_{e_1}\tilde{u} + i\tilde{\nabla}_{e_2}\tilde{u}|^2 \\
+ \left|\varepsilon d\tilde{\alpha}(e_1,e_2) - \frac{1-|\tilde{u}|^2}{2\varepsilon}\right|^2 \\
=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}
$$

We start by bounding II: we have

$$
d\tilde{\alpha} = (1-\varphi)d\alpha + \varphi d\alpha_h + d\varphi \wedge (\alpha_h - \alpha).
$$

Hence, using the fact that $|d\varphi| \leqslant C\varepsilon^{-1/2}$, we have

$$
\mathrm{II} \leqslant C\mathbf{E}_z + C|dh|^2(z) + C\varepsilon^2 \cdot C\varepsilon^{-1} \int_{B_1^2\times\{z\}} |\alpha_h - \alpha|^2.
$$

The last term is bounded by the left-hand side of (20.6); together with (20.5), this gives the desired bound.

As for IV, we note that

$$
\frac{1-|\tilde{u}|^2}{2\varepsilon} = (1-\varphi)\frac{1-|u|^2}{2\varepsilon} + \varphi\frac{1-|u_h|^2}{2\varepsilon} + O(\varepsilon^{-1}|u-u_h|),
$$

and hence IV is the squared norm of

$$
\begin{aligned}
(1-\varphi)\left[\varepsilon d\alpha(e_1,e_2) - \frac{1-|u|^2}{2\varepsilon}\right] + \varphi\left[\varepsilon d\alpha_h(e_1,e_2) - \frac{1-|u_h|^2}{2\varepsilon}\right] \\
+ O(\sqrt{\varepsilon}|\alpha-\alpha_h|) + O(\varepsilon^{-1}|u-u_h|).
\end{aligned}
$$

Thus, we have

$$\mathrm{IV} \leqslant C\mathbf{E}_z + C\varepsilon \int_{B_1^2\times\{z\}} |\alpha-\alpha_h|^2 + C\varepsilon^{-2}\int_{B_1^2\times\{z\}} |u-u_h|^2,$$

again bounded by (20.5) and (20.6).

We finally turn to I; the bound for III is obtained in the same way and hence will be skipped. We note that

$$\begin{aligned}\tilde{\nabla}\tilde{u} &= d[(1-\varphi)u+\varphi u_h] - i[(1-\varphi)u+\varphi u_h][(1-\varphi)\alpha+\varphi\alpha_h] \\ &= (1-\varphi)\nabla u + \varphi\nabla_h u_h + (u_h-u)d\varphi + O(|u-u_h||\alpha-\alpha_h|).\end{aligned}$$

Hence,

$$\begin{aligned}\int_{B_1^2\times\{z\}} |\tilde{\nabla}_{e_j}\tilde{u}|^2 \leqslant C\mathbf{E}_z + C|dh|^2(z) + C\varepsilon^{-1}\int_{B_1^2\times\{z\}} |u-u_h|^2 \\ + C\int_{B_1^2\times\{z\}} |\alpha-\alpha_h|^2.\end{aligned}$$

Again, the last two terms are bounded by the left-hand side of (20.6). This completes the proof of (20.7), and hence the proof of the proposition. □

20.2 Proof of Theorem 10.3.8

In this section we finish the proof of the stronger decay of excess for minimizers. We rescale $B_1^n(0)$ to $B_2^n(0)$ and apply Proposition 20.1.1, with some $s \in (0, \frac{1}{2})$ and $\nu > 0$ to be chosen later and with $\beta+1$ in place of β. We obtain that either $\mathbf{E} = \mathbf{E}(u, \nabla, B_2^n(0), \mathbb{R}^{n-2}) \leqslant C\varepsilon^{\beta+1}$ or the conclusions of Proposition 20.1.1 hold true (provided that the picture is rotated in such a way that $\mathbf{E}$ is small enough).

In the first situation, we clearly have $\min_S \mathbf{E}(u, \nabla, B_2^n, S) \leqslant \varepsilon^\beta$ for ε small enough and we are done. Hence, in the sequel, we can assume that we are in the second situation.

We will assume for simplicity that $\mathbb{R}^{n-2}$ minimizes $\mathbf{E}(u, \nabla, B_2^n, \cdot)$ and that

$$|dw(0)| \leqslant \delta \tag{20.1}$$

with $\delta > 0$ to be chosen momentarily.

Since $|dw(z) - dw(0)| \leqslant s \sup_{B_s^{n-2}} |D^2 w|$ on B_s^{n-2}, we have

$$\int_{B_s^{n-2}} |dw|^2 \leqslant C(n) s^{n-2} \delta^2 + C(n) s^n \sup_{B_s^{n-2}} |D^2 w|^2 \leqslant C(n) s^{n-2} (\delta^2 + s^2),$$

where the last inequality comes from the bound $\|dw\|_{L^2} \leqslant C(n)$ and standard elliptic estimates.

By item (iii) from Proposition 20.1.1 we then have

$$\int_{B_s^{n-2}} \mathbf{E}_z \leqslant \mathbf{E} \int_{B_s^{n-2}} \frac{|dw|^2}{2} + \nu \mathbf{E} \leqslant C(n) s^{n-2} (\delta^2 + s^2 + s^{2-n} \nu) \mathbf{E}.$$

This immediately gives

$$\mathbf{E}(u, \nabla, B_s^n, \mathbb{R}^{n-2}) \leqslant C(n)(\delta^2 + s^2 + s^{2-n}\nu) \mathbf{E}.$$

The theorem follows under the assumption (20.1) by taking δ, s and *subsequently* ν small enough. The general case can be reduced to this one by the very same argument of Section 17.2; the only differences here are that we use item (ii) from Proposition 20.1.1 in order to bound

$$\|h - (h)_{B_1^{n-2}} - \sqrt{\mathbf{E}} w\|_{L^2}^2 \leqslant \nu \mathbf{E}$$

and that $\mathbf{E}_1$ is replaced by $\mathbf{E}$ throughout that argument.

20.3 Proof of Theorem 10.3.7: the case of minimizers

We prove the following theorem, which contains the second part of Theorem 10.3.7.

Theorem 20.3.1. *For $n \geq 2$, there exists $\tau_0(n) > 0$ such that the following holds. If (u, ∇) is an entire, local minimizer for the energy E_1, with $u(0) = 0$ and the energy bound*

$$\frac{1}{|B_R^{n-2}|} \int_{B_R^n} \left[|\nabla u|^2 + |F_\nabla|^2 + \frac{1}{4}(1 - |u|^2)^2 \right] \leq 2\pi + \tau_0,$$

then (u, ∇) is two-dimensional. More precisely, we have $(u, \nabla) = P^*(u_0, \nabla_0)$ up to a change of gauge, where P is the orthogonal projection onto a two-dimensional subspace and (u_0, ∇_0) is the standard degree-one solution of Taubes [74] (or its conjugate), centered at the origin.

Proof. We can assume $n \geqslant 3$. We proceed exactly as in the proof of Theorem 10.3.7: letting $\beta := n - 2 > 0$, it is enough to prove that

$$\lim_{R \to \infty} R^\beta \min_S \mathbf{E}_1(u, \nabla, B_R^n, S) = 0.$$

This follows from the stronger excess decay statement for minimizers, using the same iteration argument employed in the proof of Theorem 10.3.7. □

Appendices

Appendix A

Smooth perturbation of complex polynomials

Lemma A.0.1. *For any integer $N > 0$ there exists constants $\Lambda_N > 1$ and $\varepsilon_N > 0$ with the following property: Let $P(z) = \Pi_{k=1}^{N}(z - a_i)$ be a complex polynomial with degree N with $a_1, \ldots, a_N \in B_{\frac{1}{2}}(0)$. Then for any perturbation $R : B_1(0) \to \mathbb{C}$ with $\|R\|_{C^N(B_1(0))} \leq \varepsilon_N$ there exists another complex polynomial $Q(z) = \Pi_{k=1}^{N}(z - b_j)$ with $b_1, \ldots, b_N \in B_{\frac{2}{3}}(0)$ such that:*

$$\Lambda_N^{-1} \leq \frac{|P(z) + R(z)|}{|Q(z)|} \leq \Lambda_N \,.$$

Proof. We prove the lemma by induction. By the bound $|R(z)| \leq \varepsilon_N \leq |P(z)|$ on ∂B_1 and Rouche's theorem, there exists a point $a \in B_1$ such that $P(a) + R(a) = 0$. Then we define $\tilde{R}(z) = R(z) + P(a)$ so that $\tilde{R}(a) = 0$ and we write:

$$\frac{P(z) + R(z)}{z - a} = \frac{P(z) - P(a) + \tilde{R}(z)}{z - a} = \frac{P(z) - P(a)}{z - a} + \frac{\tilde{R}(z)}{z - a} \,.$$

Since $\tilde{R}(z) = R(z) - R(a)$ we get that $\|\tilde{R}\|_{C^N(B_1)} \leq 2\varepsilon_N$. Then by a Taylor expansion we estimate:

$$\|\frac{\tilde{R}(z)}{z-a}\|_{C^{N-1}(B_1)} \leq C_N \|\tilde{R}\|_{C^N(B_1)} \leq C_N \varepsilon_N \, .$$

Now since $\frac{P(z)-P(a)}{z-a}$ is a complex polynomial with degree $N-1$, by induction we see that there exists a complex polynomial $\tilde{Q}(z)$ such that:

$$\Lambda_{N-1}^{-1} \leq \Big|\frac{P(z) - P(a) + \tilde{R}(z)}{\tilde{Q}(z)(z-a)}\Big| \leq \Lambda_{N-1} \, .$$

Naming $Q(z) = \tilde{Q}(z)(z-a)$ we get that:

$$\Lambda_N^{-1} \leq \frac{|P(z) + R(z)|}{|Q(z)|} \leq \Lambda_N \, .$$

The case $N = 1$ follows by a standard transversality argument. $\square$

Appendix B

Barycenter and variance of good slices

We show two lemmas which give a more refined control of a critical pair on a *good slice* $B_1^2 \times \{z\}$, with $z \in \mathcal{G}^\eta$, the good set defined in (15.1). We assume that (u, ∇) is a critical pair for E_ε, defined on $B_1^2 \times B_1^{n-2}$, with $\varepsilon \leqslant \varepsilon_0$ and

$$E_\varepsilon(u, \nabla) \leqslant |B_1^{n-2}|(2\pi + \tau_0)$$

(as well as (13.1)–(13.2)). Under this assumption, we have

$$\int_{(B_{3/4}^2 \setminus B_{1/2}^2) \times \{z\}} e_\varepsilon(u, \nabla) \leqslant e^{-K(n)/\varepsilon}$$

for $z \in B_{3/4}^{n-2}$, since this part of the slice is far from the vorticity set. Recall that the barycenter

$$h(z) = \Phi_{\chi(x_1, x_2)}(z)$$

was defined using a cut-off function χ supported in $B_{3/4}^2$, with $\chi = 1$ on $B_{1/2}^2$ (the notation in the subscript means $\chi \cdot (x_1, x_2)$).

Lemma B.0.1 (Barycenter of a good slice)**.** *For $\varepsilon_0, \tau_0, \eta_0 > 0$ small enough, if $\eta \leq \eta_0$ and $z \in \mathcal{G}^\eta$,* *then we have the following estimate (for a possibly different $K = K(n)$):*

$$|h(z) - h_0(z)| \le C(n)\varepsilon |\log(\boldsymbol{E}_2)_z|(\boldsymbol{E}_2)_z^{1/2} + e^{-K/\varepsilon},$$

where h_0 is the map from Proposition 15.3.1 giving the zero set on good slices.

In other words, the barycenter of the good slice is close to the actual zero of u here (unique in $B^2_{1/2} \times \{z\}$).

Proof. Recall that, by definition, we have

$$h(z) = \Phi_{\chi(x_1,x_2)}(z) = \frac{1}{2\pi}\int_{B^2_{1/2}\times\{z\}} (x_1, x_2)J(u, \nabla)(e_1, e_2) + Ce^{-K/\varepsilon}.$$

Since the integral of the Jacobian on $B^2_{1/2} \times \{z\}$ is $2\pi + O(e^{-K/\varepsilon})$ (see, e.g., the proof of [62, Lemma 6.11]), we get

$$h(z) - h_0(z) = \frac{1}{2\pi}\int_{B^2_{1/2}\times\{z\}} [(x_1, x_2) - h_0(z)]J(u, \nabla)(e_1, e_2) + Ce^{-K/\varepsilon}.$$

On the other hand, using the notation from Proposition 19.1.1, we have

$$\left|\frac{1}{2\pi}\int_{B^2_{1/2}\times\{z\}} [(x_1, x_2) - h_0(z)]J(u_{h_0}, \nabla_{h_0})(e_1, e_2)\right| \leqslant Ce^{-K/\varepsilon},$$

by symmetry of the standard planar solution.

Moreover, $u(\cdot, z)$ vanishes linearly at $h_0(z)$, as observed in Lemma 14.3.5. We can then apply a rescaling of (13.4) in Theorem 13.4.2, which gives

$$\int_{B^2_{1/2}\times\{z\}} |J(u, \nabla)(e_1, e_2) - J(u_{h_0}, \nabla_{h_0})(e_1, e_2)| \le C\sqrt{(\mathbf{E}_2)_z} + Ce^{-K/\varepsilon}. \tag{B.1}$$

Selecting a radius $C(n)\varepsilon \leqslant r \leqslant \frac{1}{4}$, we have

$$e_\varepsilon(u,\nabla)(y,z) \leqslant C(n)\varepsilon^{-2}e^{-K|y-h_0(z)|/\varepsilon} \quad \text{on } [B^2_{1/2} \setminus B^2_r(h_0(z))] \times \{z\}$$

for a possibly different K, since as observed in Lemma 14.3.5 the distance from the vorticity set Z is comparable to the distance from

$$Z \cap (B^2_{3/4} \times \{z\}) \subseteq B^2_{C(n)\varepsilon}(h_0(z)) \times \{z\},$$

on good slices. Hence,

$$\begin{aligned}
&\int_{B^2_{1/2}\times\{z\}} |(x_1,x_2)-h_0(z)||J(u,\nabla)(e_1,e_2)-J(u_{h_0},\nabla_{h_0})(e_1,e_2)| \\
&\leq r\int_{B^2_r(h_0(z))\times\{z\}} |J(u,\nabla)(e_1,e_2)-J(u_{h_0},\nabla_{h_0})(e_1,e_2)| \\
&\quad + C\varepsilon^{-2}\int_{B^2_{1/2}\setminus B^2_r(h_0(z))} |y-h_0(z)|e^{-K|y-h_0(z)|/\varepsilon}\,dy \\
&\leqslant Cr\sqrt{\mathbf{E}_z} + C\varepsilon e^{-Kr/\varepsilon}
\end{aligned}$$

for a possibly different K. Taking $r := M\varepsilon|\log \mathbf{E}_z|$ for big enough M, we get

$$r\sqrt{\mathbf{E}_z} + \varepsilon e^{-Kr/\varepsilon} \leqslant M\varepsilon\sqrt{\mathbf{E}_z}|\log \mathbf{E}_z| + \varepsilon\sqrt{\mathbf{E}_z} \leqslant C(n)\varepsilon\sqrt{\mathbf{E}_z}|\log \mathbf{E}_z|$$

(recall that $\mathbf{E}_z \leqslant \frac{1}{2}$, by definition of good set), unless $r < C(n)\varepsilon$ or $r > \frac{1}{4}$. The situation $r < C(n)\varepsilon$ cannot happen, once M is taken large enough, while in the last case we obtain $\mathbf{E}_z \leqslant e^{-K'/\varepsilon}$ and thus we are done again, by taking $r := \frac{1}{4}$ above. □

Next we show that on a good slice the variance is close to $\varepsilon^2 v_0$, where v_0 is the variance of the standard degree-one planar solution.

Lemma B.0.2 (Variance of a good slice). *For any $\sigma \in (\varepsilon, 1)$ such that $|h_0(z)| \leqslant \sigma$, we have the following estimate on good slices, for any $c \in \mathbb{R}^2$ with $|c| \leqslant \sigma$:*

$$\left| \frac{1}{2\pi} \int_{B^2_{1/2}\times\{z\}} [(x_1 - c_1)^2 + (x_2 - c_2)^2] e_\varepsilon(u, \nabla) - \varepsilon^2 v_0 \right|$$
$$\leqslant C(n)\varepsilon^2 |\log(\boldsymbol{E}_2)_z|^2 \sqrt{(\boldsymbol{E}_2)_z} + C(n)\sigma^2(\boldsymbol{E}_1)_z + C(n)|h(z) - c|^2$$
$$+ C(n)e^{-K\sigma/\varepsilon},$$

for a possibly different $K = K(n)$.

Proof. First of all, since the integrand in the definition of excess $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ upper bounds $e_\varepsilon(u, \nabla) - J(u, \nabla)$, we can replace $e_\varepsilon(u, \nabla)$ with $J(u, \nabla)$, up to an error bounded as follows: for $\mathbf{E}_1$, we bound separately the contribution of $B^2_{2\sigma}$ and the complement (where we use exponential decay) obtaining the error

$$9\sigma^2(\mathbf{E}_1)_z + C(n)e^{-K\sigma/\varepsilon};$$

as for $\mathbf{E}_2$, we argue as in the previous proof, obtaining the error

$$C(n)|h_0(z) - c|^2(\mathbf{E}_2)_z + C(n)\varepsilon^2|\log(\mathbf{E}_2)_z|^2(\mathbf{E}_2)_z + e^{-K/\varepsilon},$$

where the first term comes from replacing c with the actual location $h_0(z)$ of the zero. Moreover, by definition of v_0, we have

$$\frac{1}{2\pi}\int_{B^2_{1/2}\times\{z\}} |(x_1, x_2) - h_0(z)|^2 J(u_{h_0}, \nabla_{h_0})(e_1, e_2) = \varepsilon^2 v_0 + O(e^{-K/\varepsilon});$$

since

$$
\begin{aligned}
|(x_1, x_2) - h_0(z)|^2 - |(x_1, x_2) - c|^2 = 2\langle (x_1, x_2) - h_0(z), c - h_0(z)\rangle \\
- |c - h_0(z)|^2
\end{aligned}
$$

and

$$
\int_{B^2_{1/2}\times\{z\}} [(x_1, x_2) - h_0(z)] J(u_{h_0}, \nabla_{h_0}) = O(e^{-K/\varepsilon}),
$$

we obtain

$$
\frac{1}{2\pi}\int_{B^2_{1/2}\times\{z\}} |(x_1, x_2) - c|^2 J(u_{h_0}, \nabla_{h_0})(e_1, e_2) = |h_0(z) - c|^2 + \varepsilon^2 v_0 + O(e^{-K/\varepsilon}).
$$

As in the previous proof, we can replace $J(u_0, \nabla_0)$ with $J(u, \nabla)$ here, up to an error of the form $C(n)(|h_0(z) - c|^2 + \varepsilon^2 |\log \mathbf{E}_z|^2)\sqrt{(\mathbf{E}_2)_z}$ (using (13.4) from Theorem 13.4.2). Finally, we can bound

$$
\begin{aligned}
|h_0(z) - c|^2 &\leqslant 2|h_0(z) - h(z)|^2 + 2|h(z) - c|^2 \\
&\leqslant 2|h(z) - c|^2 + C(n)\varepsilon^2 |\log(\mathbf{E}_2)_z|^2 (\mathbf{E}_2)_z
\end{aligned}
$$

using the previous proposition, and the claim follows. ◻

Remark B.0.3. Since $t \mapsto t|\log t|^2$ is concave for $t > 0$ small enough, we have

$$
\int_S \mathbf{E}_z |\log \mathbf{E}_z|^2 \leqslant \left(\int_S \mathbf{E}_z\right)\left|\log\left(⨍_S \mathbf{E}_z\right)\right|^2 \leqslant C\left(\int_S \mathbf{E}_z\right)\left|\log\left(⨍_S \mathbf{E}_z\right)\right|^2
$$

for sets $S \subseteq \mathcal{G}^\eta$ of measure comparable with 1.

Appendix C

Poincaré–Gaffney-type inequalities

In the construction of the interpolation gauge in Proposition 19.2.1 we make frequent use of Poincaré-type inequalities for functions and differential forms. These inequalities are well known; we present the special cases used in this thesis for the convenience of the reader.

The following lemma is a consequence of results first appeared in the original paper of Gaffney [40] (see also [50] for a systematic treatment on manifolds with boundary), but for our application we need it to hold uniformly for cylinders of the form $B_1^2 \times B_r^{n-2}$ of arbitrarily small width $r > 0$.

Lemma C.0.1 (Poincaré–Gaffney-type inequality for thin cylinders). *Given a 1-form $\alpha \in \Omega^1(\overline{B}_1^2 \times \overline{B}_r^{n-2})$ with $r \leq 1$ and the Neumann boundary condition $\alpha(\nu) = 0$ on $\partial(B_1^2 \times B_r^{n-2})$, the following inequality holds:*

$$\int_{B_1^2 \times B_r^{n-2}} |\alpha|^2 \leq C(n) \int_{B_1^2 \times B_r^{n-2}} [|d\alpha|^2 + |d^*\alpha|^2].$$

Proof. Since $B_1^2 \times B_r^{n-2}$ is a convex domain and $\iota_\nu \alpha = 0$ at its boundary, we can apply [21, Remark 9] to see that

$$\int_{B_1^2 \times B_r^{n-2}} |\nabla \alpha|^2 \leq \int_{B_1^2 \times B_r^{n-2}} [|d\alpha|^2 + |d^*\alpha|^2].$$

Now we rescale the domain with the map $\phi : B_1^2 \times B_1^{n-2} \to B_1^2 \times B_r^{n-2}$ given by

$$\phi(x_1, \ldots, x_n) := (x_1, x_2, rx_3, \ldots, rx_n),$$

and define $\tilde{\alpha}(x) := \alpha(\phi(x))$ (notice that this is different from the pullback $\phi^*(\alpha)$). Then we claim that there exists a constant $C(n) > 0$ such that

$$\int_{B_1^2 \times B_1^{n-2}} |\tilde{\alpha}|^2 \leq C(n) \int_{B_1^2 \times B_1^{n-2}} |\nabla \tilde{\alpha}|^2 . \tag{C.1}$$

We prove this by compactness and contradiction. By homogeneity, suppose there exists a sequence $\tilde{\alpha}_k$ with $\iota_\nu \tilde{\alpha}_k = 0$ on $\partial(B_1^2 \times B_1^{n-2})$ and

$$\int_{B_1^2 \times B_1^{n-2}} |\tilde{\alpha}_k|^2 = 1, \quad \lim_{k \to \infty} \int_{B_1^2 \times B_1^{n-2}} |\nabla \tilde{\alpha}_k|^2 = 0.$$

Note that by the display above we have the bound $\|\tilde{\alpha}_k\|_{W^{1,2}(B_1^2 \times B_1^{n-2})} \leq 2$ for all large $k \geq 0$. Up to extracting a subsequence, we can assume that $\tilde{\alpha}_k$ converges weakly to $\tilde{\alpha}_\infty$ in $W^{1,2}$. By Rellich–Kondrachov, the convergence is strong in L^2. Thus,

$$\int_{B_1^2 \times B_1^{n-2}} |\tilde{\alpha}_\infty|^2 = 1, \quad \nabla \tilde{\alpha}_\infty = 0.$$

Hence, $\tilde{\alpha}_\infty = v$ is a constant covector. The boundary condition passes to the limit, giving that $v(\nu) = 0$ on $\partial(B_1^2 \times B_1^{n-2})$; since the normal vectors to the boundary of this domain span all of $\mathbb{R}^n$, we get that $v = 0$, a contradiction establishing (C.1). Then we compute that

$$\begin{aligned}\int_{B_1^2 \times B_r^{n-2}} |\alpha|^2 &= r^{n-2} \int_{B_1^2 \times B_1^{n-2}} |\tilde{\alpha}|^2 \\ &\leq C(n) r^{n-2} \int_{B_1^2 \times B_1^{n-2}} |\nabla \tilde{\alpha}|^2 \\ &\leq C(n) \int_{B_1^2 \times B_r^{n-2}} |\nabla \alpha|^2 \\ &\leq C(n) \int_{B_1^2 \times B_r^{n-2}} [|d\alpha|^2 + |d^*\alpha|^2],\end{aligned}$$

as desired. □

The next lemma is a weighted Poincaré estimate for functions in two dimensions.

Lemma C.0.2. *There exists a constant $C > 0$ such that for any compactly supported function $f \in C_c^1(B_R^2)$ the following weighted Poincare type estimate holds:*

$$\int_{B_R^2} |x|^2 |f|^2(x) \leq C R^{3/2} \int_{B_R^2} |x|^{5/2} |df|^2(x).$$

Proof. By scaling the domain, we can assume that $R = 1$. Then by [15, eq. (1.4)] (for the choice of constants $\alpha := 5/4$, $a := 1$, $p = q = r := 2$, $\gamma, \sigma := 1/4$) we can see that

$$\int_{B_1^2} |x|^2 |f|^2(x) \leq \int_{B_1^2} |x|^{1/2} |f|^2(x) \leq C \int_{B_1^2} |x|^{5/2} |df|^2(x).$$

This is indeed the desired conclusion. □

Lemma C.0.3 (Poincaré inequality on a thin annulus)**.** *Given $a > b \geqslant c \geqslant 0$, there exists a constant $C(n, a, b, c) > 0$ with the following property. Let f be a function in $W^{1,2}((B_a^2 \backslash B_c^2) \times \Omega)$, where $\Omega \subseteq \mathbb{R}^{n-2}$ is a convex bounded domain, such that*

$$\int_{(B_a^2 \backslash B_b^2) \times \Omega} f = 0.$$

Then the following Poincaré inequality holds:

$$\int_{(B^2_a \backslash B^2_c)\times\Omega} |f|^2 \le C(n,a,b,c)\,\mathrm{diam}(\Omega)^2 \int_{(B^2_a \backslash B^2_c)\times\Omega} |df|^2.$$

Proof. First we apply the standard Poincaré inequality on each two dimensional slice $(B^2_a \backslash B^2_c) \times \{z\}$ for any $z \in \Omega$:

$$\begin{aligned}
&\int_\Omega \left[\int_{(B^2_a \backslash B^2_c)\times\{z\}} |f|^2 \right] dz \\
&\le C(a,b,c) \int_{(B^2_a \backslash B^2_c)\times\Omega} |df|^2 + \int_\Omega \left| \int_{(B^2_c \backslash B^2_b)\times\{z\}} f \right|^2 dz.
\end{aligned}$$

Notice that the function $g(z) := \int_{(B^2_a \backslash B^2_b)\times\{z\}} f$ has zero average on Ω. Hence we can apply the Poincaré inequality on Ω to see that

$$\int_\Omega |g|^2 \leqslant C(n)\,\mathrm{diam}(\Omega)^2 \int_\Omega |dg|^2.$$

Indeed, it is well-known that the Poincaré inequality on a convex domain holds with a constant depending only on its diameter and n. This yields the desired conclusion. □

Remark C.0.4. The same conclusion holds if we assume that

$$\int_{(B^2_a \backslash B^2_b)\times\Omega'} f = 0$$

for some Ω' with $|\Omega'| \geqslant \alpha|\Omega|$ (the constant depending also on α).

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